SUGGESTED SOLUTIONS (ODD)

CHAPTER 5

NOTE: Use three-digit precision for all calculations unless otherwise stated or implied.

5-1. ***Transmission on a Slant Path.*** If the percentage transmission of a beam of light straight up through a rural aerosol is 75%, what will be the percentage transmission of the beam through the same aerosol at an angle of 45°?

SUGGESTED SOLUTION: Given $\tau_{ATM}(0°) = 0.75$, then ~

$$\tau_{ATM}(45°) = \left[\tau_{ATM}(0°)\right]^{\sec 45°} = (0.75)^{1.41} \approx 0.67$$

where $\sec 45° = \dfrac{1}{\cos 45°} = \dfrac{1}{1/\sqrt{2}} = \sqrt{2} \approx 1.41$

5-3. ***Visibility Through the Fog.*** You are driving to work through a rather thick fog in the early morning (the sun isn't quite up yet). You know you are coming to an intersection in the next half-mile where there is a traffic light. Which color light will you be able to see at the greatest distance? Which one at the closest distance? [Note: your choices are red ($\lambda \approx 0.650$ µm), yellow ($\lambda \approx 0.590$ µm), and green ($\lambda \approx 0.550$ µm).]

SUGGESTED SOLUTION: We must presume that we're dealing with Rayleigh scattering through a mist of fine particles (5 – 50 µm) because it's the only regime that has a wavelength dependence that's tractable to us. (The Mie region is too complicated, and the optical regime has no wavelength dependence.) Therefore, we're working with scattering cross-sections that we will take to be $\sigma = K/\lambda^4$ where K is a constant for the interaction of electromagnetic waves with water droplets. Peering out through our windshield, we will then suppose our visual acuity is the same for all three colors, and the minimum transmission we need through the fog to see any color of light is τ_{MIN} (often taken to be ≈ 2%). We thus require $\tau_{MIN} = e^{-n\sigma\Delta s} = e^{-n\frac{K}{\lambda^4}\Delta s_{MAX}}$ Solving for the distance, we have ~

$$\Delta s_{MAX} = \left(\frac{-\ln(\tau_{MIN})}{nK}\right)\lambda^4$$

Thus we see that we should be able to see the longest wavelength (red) first, and the shortest (green) last.

5-5. ***More using Beer's Law.*** In addition to the molecules in the last problem (5-4), suppose there is also a particulate with a uniform density distribution of $n_p = 3.00 \times 10^{14}$ cm^{-3} from sea level to 20 km altitude having a scattering cross-section of $\sigma_p = 10^{-21}$ cm^2 for $\lambda = 0.450$ µm light. Calculate the vertical transmission to space through this two-component aerosol.

SUGGESTED SOLUTION: Because the density of scatterers is uniform, transmission through their layer is $\tau_p = e^{-n_p \sigma_p \Delta s_p} = e^{-(3\times10^{20}\,m^{-3})(10^{-25}\,m^2)(2\times10^4\,m)} \approx 0.549$. Combining this with our last answer from the previous problem, we have that the total transmission through this two-component atmosphere is ~

$$\tau_2 = \tau_{0.45}\tau_p = (0.670)(0.549) \approx 0.368$$

5-7. ***Multiple Component Atmosphere.*** Use the following model of atmospheric aerosols to calculate the atmospheric transmission factor along an 8.00 km horizontal path at wavelengths of 0.500 µm, 1.00 µm, 5.00 µm, and 10.0 µm:

Particle	Diameter	Density
Smoke	0.05 µm	10^{11} m^{-3}
Fumes	0.5 µm	10^{9} m^{-3}
Dust	5 µm	10^{5} m^{-3}
Ash	50 µm	10^{3} m^{-3}

☞ See spreadsheet Chapter 05 ~ Suggested Solutions (Odd).xlsx for solution. ☜

5-9. ***Application of Beer's Law.*** The Earth receives 485 W/m^2 in the visible on its surface when the sun is directly overhead. The atmospheric transmission factor is 0.900. What is the irradiance on the surface when the sun is at a zenith angle of 30°?

SUGGESTED SOLUTION: Given the irradiance at the surface at normal incidence, $E_S(0°)$, and the vertical atmospheric transmission factor, $\tau_{ATM}(0°)$, we can deduce the irradiance above the atmosphere from the sun, E_0, as $E_0 = \dfrac{E_S(0°)}{\tau_{ATM}(0°)} = \dfrac{485\,\mathrm{W/m^2}}{0.9} \approx 539\,\mathrm{W/m^2}$.

Meanwhile, we can calculate the atmospheric transmission function when the sunlight comes in from a 30° angle: $\tau_{ATM}(30°) = \left[\tau_{ATM}(0°)\right]^{\sec 30°} = \left[0.9\right]^{1.15} \approx 0.885$.

Together with the incident irradiance from the sun, this gives a new irradiance at the Earth's surface at a 30° zenith angle, $E_S(30°)$, of

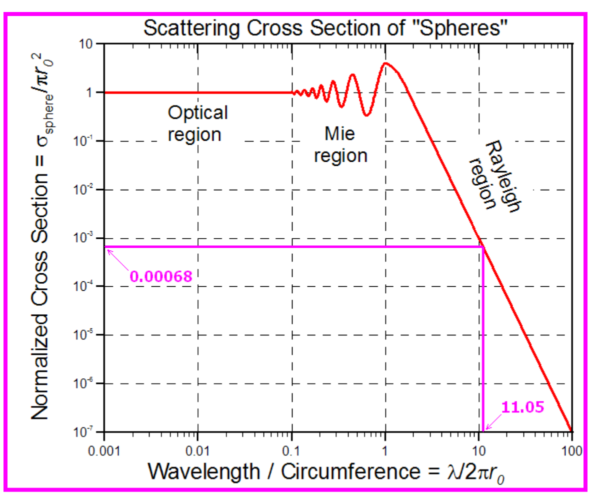
$$E_S(30°) = E_0\tau_{ATM}(30°) = (539\,\mathrm{W/m^2})(0.885) \approx 477\,\mathrm{W/m^2}$$

But wait (oh no, not again!)! This is the irradiance at Earth's surface on a surface perpendicular to the sun's rays. To calculate the irradiance on the ground, we need to correct for the zenith angle, which effectively spreads the power out into a larger area:

$$E_{ON\,GROUND}(30°) = E_S(30°)\cos 30° = (477\,\mathrm{W/m^2})(0.866) \approx 413\,\mathrm{W/m^2}$$

5-11. ***Laser Sounding Experiment.*** Return to the "laser radar" problem (4-22) and drawing in that problem set. The thin dust clouds mentioned float around in the stratosphere at an altitude of about 40 km, and are thought to result from meteorite disintegration; the average particle size is around 0.02 μm. If we shoot our ruby laser (λ = 0.6943 μm) into the air and time the duration of the echo, we can estimate that the dust layer is only about 500 m thick. In 1965, Goyer, Watson, Evans, and Gearhart from the National Center for Atmospheric Research (NCAR) did this experiment using a ten joule per pulse laser and an old surplus searchlight mirror about 2.00 m in diameter. They measured, on the average, 8.75×10^4 photons per laser pulse (one millisecond pulses). What was the number density of the dust cloud? (Hint: you can make use of the approximation $e^{-x} \approx 1 - x$ when $x \ll 1$; but after using this approximation, you need to verify that it is valid.)

SUGGESTED SOLUTION: We'll start this solution by noting that every laser pulse we shoot into the sky has about $N_0 = \frac{E_{PULSE}}{hc/\lambda}$ photons. Out of that number, we presume that around $N_{TRANS} = N_0 \tau_{DUST}$ of them are transmitted through the dust cloud (per pulse). That leaves $N_{SCAT} = N_0 - N_{TRANS} = N_0(1 - \tau_{DUST})$ to be scattered by the dust cloud. Checking the parameter $\lambda/2\pi r_0 = 0.6943\,\mu m/(2\pi)(0.01\,\mu m) \approx 11.05$, we see that our dust cloud scattering falls into the Rayleigh regime (which we also confirm by noting our estimated particle size falls in the middle of the "Rayleigh Particle" range in the figure in the last problem). Accordingly, the phase function (directions our photons are scattered into) for Rayleigh scattering is $\frac{3}{16\pi}(1 + \cos^2\theta)$, which is $\frac{3}{8\pi}$ for back-scattered ($\theta \approx 180°$) photons. Hence we will treat the small volume of the dust cloud from which laser photons are scattered as having a "photon intensity" (per pulse) of $I_{PHOTON} = \frac{3N_{SCAT}}{8\pi}$ photons/sr. The "photon irradiance" we receive back on the ground is therefore $\mathrm{E}_{PHOTON} = \frac{I_{PHOTON}}{R^2}$ photons/m^2, and the number of photons collected by our sensor is



$$N = \mathrm{E}_{PHOTON} A_R = \frac{I_{PHOTON}}{R^2}\frac{\pi D^2}{4} = \frac{3N_{SCAT}}{8\pi R^2}\frac{\pi D^2}{4}$$
$$= \frac{3N_0(1-\tau_{DUST})D^2}{32R^2} = \frac{3(1-\tau_{DUST})D^2}{32R^2}\frac{E_{PULSE}}{hc/\lambda}$$
$$= \frac{3\lambda E_{PULSE} D^2 (1-\tau_{DUST})}{32hcR^2}$$

The only term in our last expression that is not a given is $\tau_{DUST} = e^{-n_{DUST}\sigma_{DUST}\Delta s_{DUST}}$. We will take the hint in the problem stem and write this as $\tau_{DUST} \approx 1 - n_{DUST}\sigma_{DUST}\Delta s_{DUST}$ where n_{DUST} is the unknown. The term $\sigma_{DUST} = \pi r_0^2 \sigma^*$ where σ^* is found to have the value 0.00068 on the Scattering Cross Section of "Spheres" chart (the entering argument was calculated above). Substituting it all in, we have ~

$$N = \frac{3\lambda E_{PULSE} D^2 \left(1-\left(1-n_{DUST}\pi r_0^2 \sigma^* \Delta s_{DUST}\right)\right)}{32hcR^2} = \frac{3\lambda E_{PULSE} D^2 n_{DUST} \pi r_0^2 \sigma^* \Delta s_{DUST}}{32hcR^2}$$

And solving for n_{DUST} ~

$$n_{DUST} = \frac{32hcNR^2}{3\lambda E_{PULSE} D^2 \pi r_0^2 \sigma^* \Delta s_{DUST}}$$
$$= \frac{(32)(6.63\times10^{-34}\,\mathrm{J\cdot s})(3\times10^{8}\,\mathrm{m/s})(8.75\times10^{4}\,\mathrm{photons})(4\times10^{4}\,\mathrm{m})^2}{(3)(0.6943\times10^{-6}\,\mathrm{m})(10\,\mathrm{J})(2\,\mathrm{m})^2(\pi)(10^{-8}\,\mathrm{m})^2(0.00068)(500\,\mathrm{m})} \approx 10^{11}\,\mathrm{m}^{-3}$$

Finally, we need to check to see if our approximation was valid ~

$$n_{DUST}\sigma_{DUST}\Delta s_{DUST} = n_{DUST}\pi r_0^2 \sigma^* \Delta s_{DUST} = (10^{11}\,\mathrm{m}^{-3})(\pi)(10^{-8}\,\mathrm{m})^2(0.00068)(500\,\mathrm{m}) \approx 10^{-5} \ll 1. \checkmark$$