APPENDIX 5-2: Mathematical Derivation of the Exponential Dependence from the Plot

As suggested in **Footnote 10**, the hydrostatic equation (**Equation 5-2**) can be mathematically derived from the plot of number density vs. altitude (left side of **Figure 5-4**).

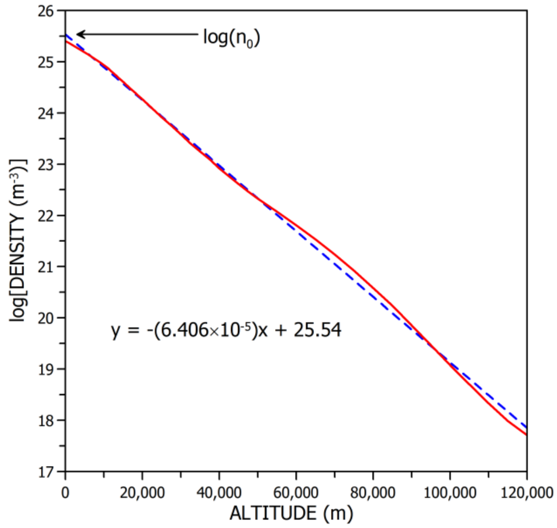

Figure 5-4A. Semi-log plot of the US Standard Atmosphere number density *vs.* altitude

First, the USSA data are replotted here (**Figure 5-4A**) with the independent variable, altitude (z), on the horizontal axis, and the logarithm of the dependent variable, number density (n), on the vertical axis. The values in the table have been converted into standard SI units for this plot: altitude in meters and number density in molecules per cubic meter. Note the value of the logarithm is shown on the vertical axis rather than the number density itself.

Next, since the empirical data (red line) appear to be nearly a straight line relationship, an approximate linear best fit (blue dashed line using Microsoft Excel's Trendline feature) of the form

$$y = mx + b \tag{1}$$

is found, where x plays the role of z and y is $\log(n)$. The coefficients $m = -6.406 \times 10^{-5}$ and $b = 25.54$ are the "slope" and "intercept," respectively, and it is convenient to call $b = \log(n_0)$. Thus the blue dashed line has the equation

$$\log(n) = mz + \log(n_0) \tag{2}$$

To solve this equation for n, follow the standard rules of algebra:

$$10^{\log(n)} = 10^{mz + \log(n_0)} = 10^{mz} 10^{\log(n_0)}, \tag{3}$$

which reveals the advertised exponential dependence:

$$n = n_0 10^{mz} \tag{4}$$

For a more familiar appearance, the exponential can be changed to the base of natural logarithms:

$$n = n_0 e^{(m \ln 10) z} \tag{5}$$

which is of the form of **Equation 5-2** provided the substitution

$$m \ln 10 = \frac{-1}{\hat{H}} \tag{6}$$

is made.

Assuming the straight line in the semi-log plot adequately approximates the number density (which is equivalent to assuming an isothermal atmosphere), this mathematical derivation gives the constants to be

$$n_0 = 10^b = 10^{25.54} \approx 3.47 \times 10^{25} \quad \left[\mathrm{m}^{-3}\right] \tag{7}$$

and

$$\hat{H} = \frac{-1}{m \ln 10} = \frac{-1}{(-6.406 \times 10^{-5})(2.303)} \approx 6780 \quad [\mathrm{m}] \tag{8}$$