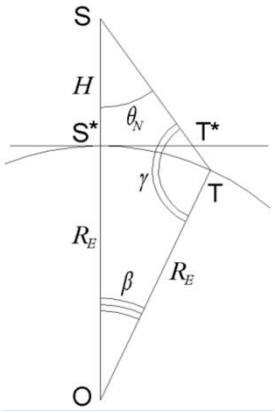
APPENDIX 4-3
High-altitude Geometry Calculation



Given H, R_E, and θ_N …

(1) **FIND** $\overline{ST^*}$ from $\triangle SS^*T^*$

SOLUTION:
$$\boxed{\overline{ST^*} = \frac{H}{\cos\theta_N}}$$

(2) **FIND** $\overline{ST}$ as the third side of $\triangle SOT$ where sides R_E and $R_E + H$ and angle θ_N are known

SOLUTION: *EITHER* …

Use the Sine Law to find[1] angle γ:

$$\frac{R_E + H}{\sin\gamma} = \frac{R_E}{\sin\theta_N} \quad \Rightarrow \quad \gamma = \sin^{-1}\left[\frac{R_E + H}{R_E}\sin\theta_N\right]$$

Then use the sum of angles in a triangle rule to find Earth center angle β:

$$\beta = 180° - \theta_N - \gamma$$

Then use the Sine Law again to find $\overline{ST}$:

$$\frac{R_E}{\sin\theta_N} = \frac{\overline{ST}}{\sin\beta} \quad \Rightarrow \quad \boxed{\overline{ST} = \frac{\sin\beta}{\sin\theta_N}R_E}$$

OR …

Use the Cosine Law:

$$R_E^2 = \left(R_E + H\right)^2 + \overline{ST}^2 - 2\left(R_E + H\right)\left(\overline{ST}\right)\cos\theta_N$$

Then use the quadratic formula:

$$\boxed{\overline{ST} = \left(R_E + H\right)\cos\theta_N - \sqrt{\left(R_E + H\right)^2\cos^2\theta_N - 2R_E H - H^2}}$$

[1] CAUTION: angle γ is greater than 90º but a computer will return the "Principal Value" of the inverse sine: $-90° \le PV \le 90°$. To find the correct value, take $\gamma = 180° - PV$.

(3) **FIND** the error in flat Earth range:

SOLUTION:

$$\text{Error} \approx \frac{\overline{\overline{ST}} - \overline{\overline{ST^*}}}{\overline{\overline{ST^*}}}$$

(4) **FIND** the maximum nadir angle when line-of-sight is tangent to Earth (i.e., when T is on the limb or horizon)

SOLUTION:

$$\theta_{N,MAX} = \sin^{-1}\left(\frac{R_E}{R_E + H}\right)$$