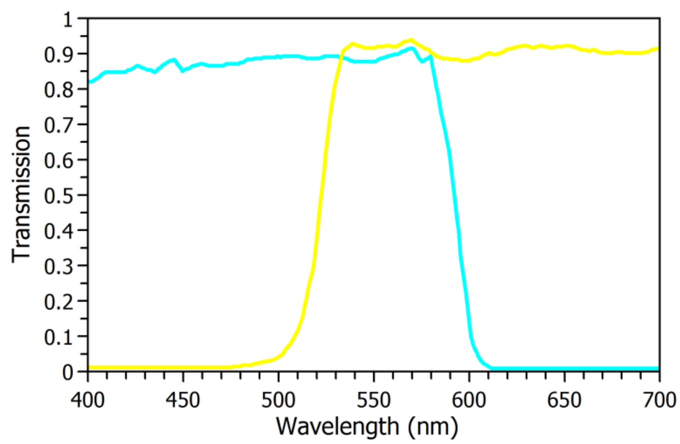
SUGGESTED SOLUTIONS (ODD)

CHAPTER 9

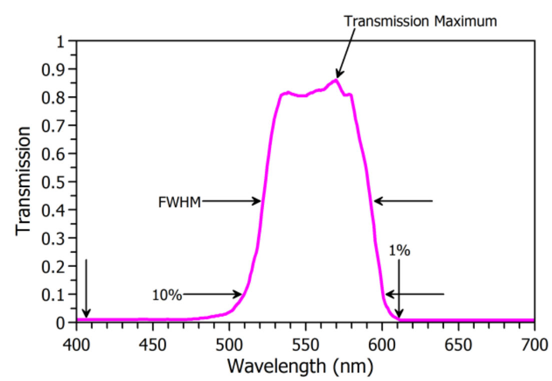
NOTE: Use three-digit precision for all calculations unless otherwise stated or implied.

9-1. *Dye Filters.* A major manufacturer of optical components can supply "subtractive" dye filters in the visible bandpass. The plot below shows the transmission of the Cyan and Yellow filters. A data sheet for the two filters is given in the companion spreadsheet. Calculate and plot the resultant effective filter, as in Figure 9-2 of the text, when the two are combined. What is the maximum transmission of the combination filter, and what are the short- and long-wavelength cutoffs at the half-maximum, 10%, and 1% points?



SUGGESTED SOLUTION: The companion solutions spreadsheet calculates the product of the two filter functions. By inspection, the maximum transmission is 86.17% at a wavelength of approximately 569.7 nm. The following table gives the short- and long-wavelength cutoffs for the three conditions specified, as well as the filter's bandpass width and center. The half-maximum width (often called Full-Width-at-Half-Maximum, FWHM) refers to one-half of the filter's actual maximum transmission, while 10% and 1% cutoffs refer to the actual 10% and 1% transmission limits, respectively. Note that there is probably considerable uncertainty as to exactly where the 1% points are, particularly on the short-wavelength side. [All values in the table are in nanometers.]

	Short-wavelength cutoff	Filter width	Filter center wavelength	Long-wavelength cutoff
FWHM	522.0	70.6	557.3	592.7
10%	510.1	90.8	555.5	600.9
1%	406.4	204.6	508.7	611.0



(This combination filter is very close to what the manufacturer would provide as a subtractive Magenta filter.)

9-3. *Algebraically Adding Two Light Waves*. Show that the two light waves in **Problem 2** algebraically added together are

$$E \approx 2.83\sin\left(\frac{7\pi t}{2} + 93.5°\right)$$

SUGGESTED SOLUTION: The answer is given in the text as **Equation 9-3**, but to *show* that it is correct, add the two waves together and postulate the answer to be a single sine wave with an unknown amplitude, A, and phase, ψ :

$$A_1 \sin(2\pi ft + \varphi_1) + A_2 \sin(2\pi ft + \varphi_2) = A\sin(2\pi ft + \psi)$$

Now use the **trig identity** $\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$ to expand the waves on both the left and the right ~

$$A_1\left[\sin(2\pi ft)\cos\varphi_1 + \cos(2\pi ft)\sin\varphi_1\right] + A_2\left[\sin(2\pi ft)\cos\varphi_2 + \cos(2\pi ft)\sin\varphi_2\right]$$
$$= A\left[\sin(2\pi ft)\cos\psi + \cos(2\pi ft)\sin\psi\right].$$

Next, regroup the terms in a more suggestive manner:

$$\sin(2\pi ft)\left[A_1 \cos\varphi_1 + A_2 \cos\varphi_2\right] + \cos(2\pi ft)\left[A_1 \sin\varphi_1 + A_2 \sin\varphi_2\right]$$
$$= \sin(2\pi ft)\left[A\cos\psi\right] + \cos(2\pi ft)\left[A\sin\psi\right].$$

For this equation to be true, the coefficients of $\sin(2\pi ft)$ and $\cos(2\pi t)$ on the left and right sides (terms in square brackets) must be equal; so write them out ~

$$\begin{aligned} A\sin\psi &= A_1 \sin\varphi_1 + A_2 \sin\varphi_2 \\ A\cos\psi &= A_1 \cos\varphi_1 + A_2 \cos\varphi_2. \end{aligned} \qquad \textbf{(9-3-1)}$$

Note this has turned one equation into two, which is the prescription necessary for solving for the two unknowns A and ψ . To solve for A, square these two resultant equations,

$$A^2 \sin^2\psi = A_1^2 \sin^2\varphi_1 + 2A_1A_2 \sin\varphi_1 \sin\varphi_2 + A_2^2 \sin^2\varphi_2$$
$$A^2 \cos^2\psi = A_1^2 \cos^2\varphi_1 + 2A_1A_2 \cos\varphi_1 \cos\varphi_2 + A_2^2 \cos^2\varphi_2,$$

add them together,

$$A^2\left[\sin^2\psi + \cos^2\psi\right]$$
$$= A_1^2\left[\sin^2\varphi_1 + \cos^2\varphi_1\right] + 2A_1A_2\left[\cos\varphi_1 \cos\varphi_2 + \sin\varphi_1 \sin\varphi_2\right] + A_2^2\left[\sin^2\varphi_2 + \cos^2\varphi_2\right],$$

and use the Pythagorean **trig identity** $\sin^2\theta + \cos^2\theta = 1$:

$$A^2 = A_1^2 + 2A_1A_2\left[\cos\varphi_1 \cos\varphi_2 + \sin\varphi_1 \sin\varphi_2\right] + A_2^2.$$

Finally, use the **trig identity** $\cos\varphi_1 \cos\varphi_2 + \sin\varphi_1 \sin\varphi_2 = \cos(\varphi_1 - \varphi_2)$:

$$A^2 = A_1^2 + 2A_1A_2 \cos(\varphi_1 - \varphi_2) + A_2^2.$$

To solve for the phase angle, ψ, the two equations **(9-3-1)** may be simply divided, and use the fundamental **trig identity** $\frac{\sin\theta}{\cos\theta}=\tan\theta$:

$$\frac{A\sin\psi}{A\cos\psi}=\tan\psi=\frac{A_1\sin\varphi_1+A_2\sin\varphi_2}{A_1\cos\varphi_1+A_2\cos\varphi_2}$$

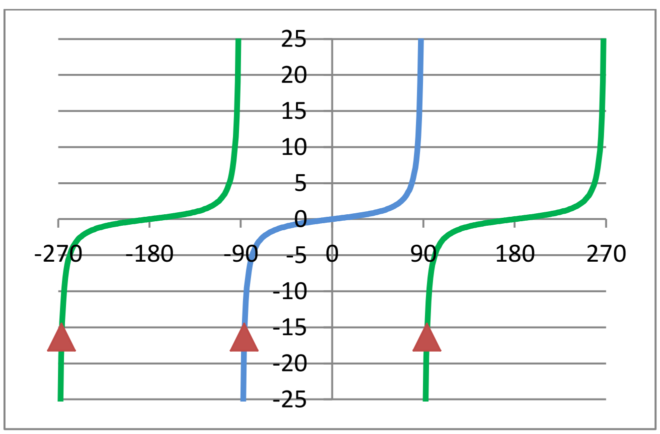
This derivation fills in the gaps for **Equation 9-3** in the text, and now we can calculate ~

$$A=\sqrt{4^2+2\cdot4\cdot3\cos\left(\frac{\pi}{4}-\pi\right)+3^2}\approx2.83,$$

which is quite straightforward, and ~

$$\psi=\tan^{-1}\left(\frac{4\sin\frac{\pi}{4}+3\sin\pi}{4\cos\frac{\pi}{4}+3\cos\pi}\right)\approx-86.5°,$$

which is **not** straightforward. The difficulty with applying the formula for phase angle is that calculators and computers always return the *Principal Value* of inverse trigonometric functions. This is the blue line in the graph of the tangent, below, where the apparent phase angle is marked with a red triangle. But with the tangent – and all periodic functions – there is the possibility that a value in another cycle of the function could be the correct one (the other red triangles). In the case of the tangent, a peek at the plot given in **Problem 2** suggests that we should choose the value on the green branch near 90°.

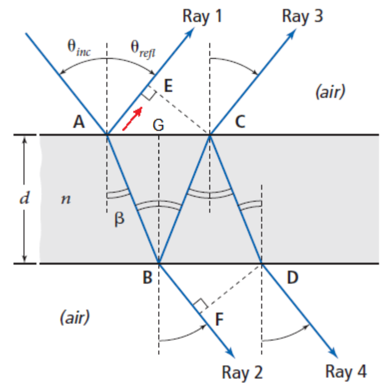


Therefore, moving to the correct branch of the tangent, $\psi=-86.5°+180°=93.5°$, we have ~

$$E\approx2.83\sin\left(\frac{7\pi t}{2}+93.5°\right),$$

but care must be taken when computing this resultant wave in making sure the argument of the sine function is either in radians or degrees (but not mixed).

9-5. ***Reflection and Transmission Criteria*****.** Referring to the simplified thin film application, **Figure 9-6** (reproduced at right and slightly modified with the addition of point G), derive the optical path length difference **Equations 9-8** and **9-11** for reflection and transmission, respectively:



$$\Gamma = 2nd\cos\beta + \frac{\lambda}{2} \qquad \textbf{(9-8)}$$

and

$$\Gamma = 2nd\cos\beta \qquad \textbf{(9-11)}$$

[HINT: You will need a trig identity or two, and the red arrow reminds you where there is a 180° phase change on reflection.]

SUGGESTED SOLUTION: For reflection, E is abeam C, and we need the difference between paths A-to-E and A-to-B-to-C. So first look at right triangle A-E-C, and note that angle A-C-E is the same as the incident angle: $\angle \text{ACE} = \theta_{inc}$. Then,

$$\overline{\text{AE}} = \overline{\text{AC}}\sin\theta_{inc} = 2\overline{\text{AG}}(n\sin\beta) = 2n\overline{\text{AG}}\sin\beta$$

where we have made use of Snell's Law (assuming $n_{air} = 1$), and have introduced the distance A-to-G which also appears in right triangle A-G-B. Now in right triangle A-G-B, we have $\overline{\text{BG}} = d$, and thus $\overline{\text{AG}} = d\tan\beta$. Substituting,

$$\overline{\text{AE}} = 2n(d\tan\beta)\sin\beta = 2nd\frac{\sin^2\beta}{\cos\beta}$$

Since $n_{air} = 1$, this physical distance is the same as the optical path distance.

Now turning our attention to path A-B-C, which equals twice path A-B, we also have in right triangle A-G-B that $\overline{\text{AB}} = \frac{\overline{\text{BG}}}{\cos\beta} = \frac{d}{\cos\beta}$. The *physical path* is thus $2\overline{\text{AB}} = \frac{2d}{\cos\beta}$, but the *optical path* is

$$2n\overline{\text{AB}} = \frac{2nd}{\cos\beta},$$

which is the measure of the path length in wavelengths reduced by the index of refraction (n) in the thin film.

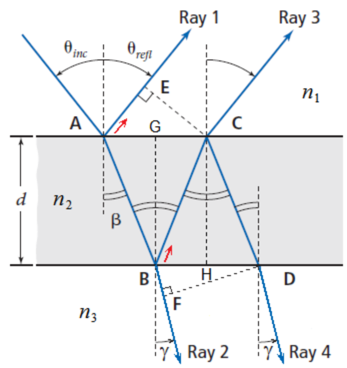
The difference between these optical path lengths is partly what we are seeking:

$$\frac{2nd}{\cos\beta} - 2nd\frac{\sin^2\beta}{\cos\beta} = 2nd\left(\frac{1}{\cos\beta} - \frac{\sin^2\beta}{\cos\beta}\right) = 2nd\left(\frac{1-\sin^2\beta}{\cos\beta}\right) = 2nd\left(\frac{\cos^2\beta}{\cos\beta}\right) = 2nd\cos\beta$$

In addition to this term, electromagnetic boundary conditions impose the stipulations that the tangential (to the interface between air and material) component of the electric field but the normal component of what is called the displacement field (a relative of the electric field) must be continuous from one side to the other. (See advanced texts in electromagnetic theory.) When the light beam passes from a medium of lower to a medium of higher index of refraction, these conditions cause the reflected portion to flip over – to have an additional 180° (or π radians) phase change, which is the same as a one-half wavelength shift in the optical path length.[1] Thus,

$$\Gamma = 2nd\cos\beta + \frac{\lambda}{2}$$

Now that the reader has seen how the difference in optical path lengths is derived for reflection, it will be left as an exercise within this solution for the reader to perform the same manipulations for transmission. It goes the same, but without the added half-twist at the end.



9-7. ***More Reflection and Transmission Criteria*.** Again refer to a modified version of **Figure 9-6** (points G and H have been added), but this time let the top, middle, and bottom materials have indices of refraction n_1, n_2, and n_3 such that $n_1 < n_2 < n_3$, like a thin "anti-reflection" coating on the lens of your eyeglasses. (Note that light rays now bend *toward* the normal at both interfaces according to Snell's Law: $\theta_{inc} > \beta > \gamma$.) Find the optical path length differences (like **Equations 9-8 and 9-11**) for Rays 1 and 3 and for Rays 2 and 4, and deduce the criteria for reflection and transmission from/through the thin film. [HINT: Red arrows remind you where the light waves have a 180° phase shift on reflection (only).]

SUGGESTED SOLUTION: First, we consider Ray 3 at C where it is abeam Ray 1 at E. The *optical path length* (measured in wavelengths) difference is

$$\begin{aligned}\Gamma_{\text{C-E}} &= opl(\text{ABC}) - opl(\text{AE}) \\ &= \left[n_2\overline{\text{AB}} + \frac{\lambda}{2} + n_2\overline{\text{BC}} \right] - \left[\frac{\lambda}{2} + n_1\overline{\text{AE}} \right] \\ &= 2n_2\overline{\text{AB}} - n_1\overline{\text{AE}}\end{aligned}$$

where "*olp*(-)" means optical path length and the overbar notation means physical path length. As in **Problem 5**, looking at right triangles A-G-B and A-E-C we have ~

$$\overline{\text{AB}} = \frac{d}{\cos\beta} \quad \text{and} \quad \overline{\text{AE}} = 2\overline{\text{AG}}\sin\theta_{inc} = 2d\tan\beta\sin\theta_{inc}$$

Substituting these expressions, doing a little rearrangement, using Snell's Law, and pulling out a couple of trig identities gives us the difference we are looking for:

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[1] Is the wave shifted 180° (or one-half wavelength) forward or backward? That is, should we use a plus or a minus sign? It doesn't make any difference because the wave is periodic, so you can't tell the difference physically. Mathematically, chose whichever sign makes the solution most convenient. In this problem, we'll chose to add a half wavelength, but in a later problem, we'll see that it makes more sense to subtract a half wavelength.

$$\begin{aligned}\Gamma_{\text{C-E}} &= 2n_2\left(\frac{d}{\cos\beta}\right) - n_1\left(2d\tan\beta\sin\theta_{inc}\right) = 2n_2\frac{d}{\cos\beta} - 2d\tan\beta\left(n_1\sin\theta_{inc}\right)\\ &= 2n_2\frac{d}{\cos\beta} - 2d\tan\beta\left(n_2\sin\beta\right) = 2n_2 d\left(\frac{1}{\cos\beta} - \frac{\sin\beta}{\cos\beta}\sin\beta\right)\\ &= 2n_2 d\left(\frac{1-\sin^2\beta}{\cos\beta}\right) = 2n_2 d\left(\frac{\cos^2\beta}{\cos\beta}\right) = 2n_2 d\cos\beta.\end{aligned}$$

From this relation, we reason that **constructive interference** (reflection) will occur when the *opl* is an integer multiple of the wavelength, but we will have **destructive interference** (no reflection) when the *opl* is a half-integer multiple. Summarizing:

$$\Gamma_{\text{C-E}} = 2n_2 d\cos\beta = \begin{cases} m\lambda & \text{for constructive interference (reflection)} \\ (m-0.5)\lambda & \text{for destructive interference (no reflection)} \end{cases}$$

where m is a positive integer, $m = 1, 2, 3, \ldots$.

Next, considering Ray 4 at D where it is abeam Ray 2 at F, the *opl* by the same math as above is:

$$\begin{aligned}\Gamma_{\text{D-F}} &= opl(\text{BCD}) - opl(\text{BF}) = \left[-\frac{\lambda}{2} + n_2\overline{\text{BC}} + n_2\overline{\text{CD}}\right] - \left[n_3\overline{\text{BF}}\right]\\ &= 2n_2\overline{\text{BC}} - n_3\overline{\text{BF}} - \frac{\lambda}{2} = 2n_2\left(\frac{d}{\cos\beta}\right) - n_3\left(2d\tan\beta\sin\gamma\right) - \frac{\lambda}{2}\\ &= 2n_2\left(\frac{d}{\cos\beta}\right) - 2d\tan\beta\left(n_3\sin\gamma\right) - \frac{\lambda}{2} = 2n_2\left(\frac{d}{\cos\beta}\right) - 2d\tan\beta\left(n_2\sin\beta\right) - \frac{\lambda}{2}\\ &= 2n_2 d\left(\frac{1}{\cos\beta} - \frac{\sin\beta}{\cos\beta}\sin\beta\right) - \frac{\lambda}{2} = 2n_2 d\left(\frac{1-\sin^2\beta}{\cos\beta}\right) - \frac{\lambda}{2}\\ &= 2n_2 d\left(\frac{\cos^2\beta}{\cos\beta}\right) - \frac{\lambda}{2} = 2n_2 d\cos\beta - \frac{\lambda}{2}.\end{aligned}$$

(We have chosen the 180° phase shift to be represented by $-\frac{\lambda}{2}$ for reasons that will become apparent later. See Footnote 1 for **Problem 5**.)

From this second relation, we also see that **constructive interference** (transmission) will occur when the *opl* is an integer multiple of the wavelength, but **destructive interference** (no transmission) will happen when the *opl* is a half-integer multiple. Summarizing:

$$\Gamma_{\text{D-F}} = 2n_2 d\cos\beta - \frac{\lambda}{2} = \begin{cases} m\lambda & \text{for constructive interference (transmission)} \\ (m-0.5)\lambda & \text{for destructive interference (no transmission).} \end{cases}$$

At first glance, the two *opl* conditions seem to be opposed to one another. However, we can reconcile them as follows: for reflection from the thin film, we want the top surface *opl* to constructively interfere upward,

$$\Gamma_{\text{C-E}} = 2n_2 d\cos\beta = m\lambda\,,$$

but we want the bottom surface *opl* to destructively interfere downward (i.e., NOT transmit through the interface), which is[2]

$$\Gamma_{\text{D-F}} = 2n_2 d\cos\beta - \frac{\lambda}{2} = (m-0.5)\lambda \ \textit{-OR-}\ 2n_2 d\cos\lambda = m\lambda$$

THESE TWO CONDTIONS ARE THE SAME: THERE IS INDEED CONSISTENCY BETWEEN THEM! Similarly for transmission through the thin film. We can therefore summarize:

For reflection: $\quad 2n_2 d\cos\beta = m\lambda$

For transmission: $\quad 2n_2 d\cos\beta = (m-0.5)\lambda.$

[2] See here why we chose to subtract a half-wavelength instead of add it. Remember it makes no difference physically, but it makes the math work out nicer.

9-9. ***Antiscratch Coatings*.** Camera lenses and eyeglasses often have a hard layer of magnesium fluoride (MgF_2, n = 1.38) on them for protection. If a typical layer is about 10 μm thick, and the glass has a nominal refractive index of 1.60, are there any wavelengths in the visible spectrum that are specifically intensified (i.e., transmission enhanced)?

SUGGESTED SOLUTION: As shown in **Problem 9**, when a thin film's substrate has a higher index of refraction, the criterion for transmission (constructive interference) is

$$2nd\cos\beta = (m-0.5)\lambda \quad \Rightarrow \quad \lambda = \frac{2nd\cos\beta}{m-0.5}$$

For most vision and photography, we suppose light is mostly normally incident, so $\cos\beta \approx 1$ again. This leaves us with ~

$$\lambda = \frac{2nd}{m-0.5} = \frac{(2)(1.38)(10\ \mu\text{m})}{m-0.5} = \frac{27.6\ \mu\text{m}}{m-0.5}$$

Examining this expression, we find that we don't get to visible wavelengths (400 – 700 μm) until around m = 40 or so. Above that, yes, some wavelengths would constructively interfere within the film and be enhanced. That's assuming the light has another property not previously addressed in this chapter: coherence. Coherence is the ability of light waves to remain in phase with one another (and hence interfere) over long distances. The requirements for coherent light are perfectly planar wavefronts and exact monochromaticity. Ordinary light from the sun and most artificial sources does not generally meet these stringent tests. (Only lasers come close, and then not over excessively long distances.) Thus, we can expect there will be NO wavelengths that are specifically intensified in passing through the protective coatings on our eyeglasses or camera lenses (as we confirm from daily experience).

9-11. ***Interference Filter*.** An interference filter using a thin film of material with index of refraction $n_F = 1.60$ is designed to have a transmission peak $\Delta\lambda_{MIN} = 5.00\ \text{nm}$ wide about a central wavelength[3] of $\lambda = 632.8\ \text{nm}$ when used at normal incidence, and a (partially silvered) reflectance of $R = 0.80$ at the material-air interface. How thick must the film be?

SUGGESTED SOLUTION: First, from the width of the transmission peak, we need to perform the following check ~

$$\Delta\lambda_{MIN} = \frac{\lambda(1-R)}{m\pi\sqrt{R}}$$

$$m = \left(\frac{\lambda}{\Delta\lambda_{MIN}}\right)\left(\frac{1-R}{\pi\sqrt{R}}\right) = \left(\frac{632.8\ \text{nm}}{5.00\ \text{nm}}\right)\left(\frac{1-0.80}{\pi\sqrt{0.80}}\right) \approx 9.01$$

That is, since we are dealing with a transmission peak, the order, m, is supposed to be an integer, which we have just confirmed. Evidently we are supposed to use m = 9 for this problem.

[3] He:Ne laser wavelength.

Proceeding, for constructive interference in the forward (transmission) direction we note that $\beta = 0°$ and calculate ~

$$2nd\cos\beta = m\lambda$$

$$d = \frac{m\lambda}{2n\cos\beta} = \frac{(9)(632.8\text{ nm})}{(2)(1.60)\cos 0°} = \frac{(9)(632.8\text{ nm})}{(2)(1.60)(1)} \approx 1780\text{ nm} = 1.78\ \mu\text{m}.$$