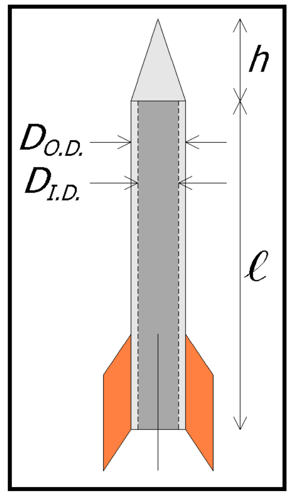
CHAPTER 13 ~ SUGGESTED SOLUTIONS (ODD)

13-1. ***A Little Rocket Science.*** Suppose that the Rocket Boys[1] of southern West Virginia construct a rocket out of a $\ell \approx 1$ m steel pipe[2] (O.D. ≈ 4" and I.D. ≈ 3.5") with a $h \approx 6$" solid steel nose cone. (Assume the steel has about the density of iron $\rho_{FE} \approx 7.68$ g/cm^3.) They fill the tube with a "rocket candy" solid fuel having $I_{SP} \approx 100$ s and $\rho_{RC} \approx 1.38$ g/cm^3. Using just Newtonian physics by neglecting air drag and ignoring any nozzle-induced pressure differential components of thrust (simple rockets like this one don't have nozzles to speak of), calculate the time from ignition to burn-out, the vertical velocity at burn-out, and their expected maximum altitude. The mass flow rate during burn is approximately 2.45 kg/s.

SUGGESTED SOLUTION Working with the volumes and masses first, but ignoring any additional mass of possible stabilizing fins ~



Fuel Volume, $V_{RC} = \dfrac{\pi D_{I.D.}^2}{4}\ell \approx 6210 \text{ cm}^3$

Fuel Mass, $m_{RC} = \rho_{RC} V_{RC} \approx 8.57 \text{ kg}$

Body Volume, $V_{RB} = \dfrac{\pi D_{O.D.}^2}{4}\ell - V_{RC} \approx 1900 \text{ cm}^3$

Body Mass, $m_{RB} = \rho_{FE} V_{RB} \approx 14.6 \text{ kg}$

Nose Volume, $V_{NC} = \dfrac{1}{3}\dfrac{\pi D_{O.D.}^2}{4} h \approx 412 \text{ cm}^3$

Nose Mass, $m_{NC} = \rho_{FE} V_{NC} \approx 3.16 \text{ kg}$

LIFT-OFF MASS, $m_0 = (m_{RB} + m_{NC}) + m_{RC} \approx 26.3 \text{ kg}$

BURN-OUT MASS, $m_{BO} = m_{RB} + m_{NC} \approx 17.8 \text{ kg}$

Now for the motion: Newton's Second Law says for the powered flight portion of the launch …

$$F = ma = m\frac{dv}{dt} = \text{Thrust} - \text{Weight} \approx \dot{m} v_e - mg$$

since we are neglecting drag, and we're assuming there is no pressure differential component to thrust. With $m = m_0 - \dot{m}t$ where $\dot{m}$ is the (constant) mass flow rate, t is time after ignition/lift-off, and taking v_e to be a constant characteristic of the motor, we have, upon integrating[3] ~

[1] Original title of the autobiographical book now known as "October Sky" by Homer Hickam, Jr. (Delacorte Press, 1998). It was dramatized in a film in 1999 that doesn't exactly follow the book. Note that Rocket Boys is an anagram of October Sky.

[2] O.D. means outside diameter and I.D. means inside diameter (of a tube or pipe). Assume the density of steel is the same as that of iron, $\rho_{FE} \approx 7.68$ g/cm^3.

[3] Thanks to the CRC Standard Math Tables, 26th Edition (CRC Press, 1981).

$$v(t) = \int_0^v dv = \int_0^t \left(\frac{\dot{m} v_e}{m_0 - \dot{m} t} - g \right) dt = -v_e \ln\left(1 - \frac{\dot{m}}{m_0} t \right) - gt \ ,$$

and assuming motion is only in the upward (z) direction[4] $\left(v(t) = \dfrac{dz(t)}{dt} \right)$ ~

$$z(t) = \int_0^z dz = \int_0^t \left[-v_e \ln\left(1 - \frac{\dot{m}}{m_0} t \right) - gt \right] dt = \frac{m_0 v_e}{\dot{m}} \left\{ \left(1 - \frac{\dot{m}}{m_0} t \right) \left[\ln\left(1 - \frac{\dot{m}}{m_0} t \right) - 1 \right] + 1 \right\} - \frac{1}{2} g t^2 .$$

Now we need v_e which we get from the specific impulse ~

$$I_{SP} = \frac{\text{Thrust}}{\dot{W}} = \frac{\dot{m} v_e}{\dot{m} g} \quad \Rightarrow \quad v_e \approx I_{SP} g \approx (100\,\text{s})(9.81\,\text{m}\cdot\text{s}^{-2}) \approx 981\,\text{m}\cdot\text{s}^{-1} .$$

We also need the rocket motor burn-out time, t_{BO}, which we get from ~

$$t_{BO} = \frac{m_{RC}}{\dot{m}} \approx \frac{8.57\text{ kg}}{2.45\text{ kg}\cdot\text{s}^{-1}} \approx 3.50\text{ s} .$$

Then noting that $\dot{m} t_{BO} = m_{RC}$, we can plug in for the burn-out speed[5] ~

$$v_{BO} \approx -(981\,\text{m}\cdot\text{s}^{-1}) \ln\left(1 - \frac{8.57\text{ kg}}{26.3\text{ kg}} \right) - (9.81\,\text{m}\cdot\text{s}^{-2})(3.50\text{ s}) \approx 352\text{ m}\cdot\text{s}^{-1} \ \text{(about Mach 1)},$$

and for the burn-out altitude …

$$z_{BO} \approx \frac{(26.3\text{kg})(981\text{m}\cdot\text{s}^{-1})}{2.45\text{ kg}\cdot\text{s}^{-1}} \left\{ \left(1 - \frac{8.57\text{kg}}{26.3\text{kg}} \right) \left[\ln\left(1 - \frac{8.57\text{kg}}{26.3\text{kg}} \right) - 1 \right] + 1 \right\} - \frac{1}{2} (9.81\text{m}\cdot\text{s}^{-2})(3.50\text{s})^2 \approx 572\,\text{m} .$$

The burn-out time, location (altitude in this case), and speed are known as the the **BURN-OUT STATE VECTOR**, (t_{BO}, z_{BO}, v_{BO}), which inserts the rocket into its gravity-only ballistic mid-course flight. (Although this model rocket probably does not go high enough to escape the influence on atmospheric drag.) At that point, the rocket body just obeys the usual kinematic rules of freshman physics (assuming its max altitude is much less than R_E so we can ignore the altitude-dependence of g), from which we can calculate the time to reach apogee and its altitude (when $v = 0$):

$$v = v_{BO} - g(t - t_{BO}) \quad \Rightarrow \quad t_{APOGEE} = \frac{v_{BO}}{g} + t_{BO} \approx \frac{352\,\text{m}\cdot\text{s}^{-1}}{9.81\text{m}\cdot\text{s}^{-2}} + 3.50\text{s} \approx 39.4\text{ s} \text{, and}$$

---

[4] Thanks again to the CRC Standard Math Tables, 26th Edition (CRC Press, 1981).

[5] Note the average acceleration is approximately $\bar{a} \approx \dfrac{\Delta v}{\Delta t} \approx \dfrac{352\,\text{m}\cdot\text{s}^{-1}}{3.50\text{s}} \approx 101\text{m}\cdot\text{s}^{-2} \approx 10.3$ “g’s”. If you were an astronaut on this rocket, it would definitely not be a soft ride.

$$v^2 = v_{BO}^2 - 2g(z - z_{BO}) \quad \Rightarrow \quad z_{APOGEE} = \frac{v_{BO}^2}{2g} + z_{BO} \approx \frac{\left(352\,\mathrm{m \cdot s^{-1}}\right)^2}{(2)\left(9.81\,\mathrm{m \cdot s^{-2}}\right)} + 572\,\mathrm{m} \approx 6{,}890\ \mathrm{m}\,.$$

This is impressive, but then this self-made model rocket is significantly larger than the sort of thing that is available through hobby stores. (Don't try this at home!)

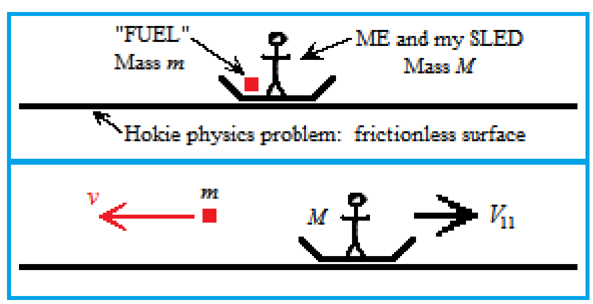
COMMENT: It is probably a serious error to neglect atmospheric drag since this simple-minded calculation produced a speed of about Mach 1 (≈ 300 m/s) without it. This would complicate the problem, however, so we won't include it. Also, the drag would play a big role in the descent from apogee ~ the rocket body may attain its terminal velocity and would take a longer time to come down than it did to go up. Yet another complication we could crank in would be the addition of a nozzle on our rocket. There would then be no closed-form solution because atmospheric parameters would have to be calculated along the trajectory.

ANOTHER COMMENT: Considerable Googling turns up mass flow rates ($\dot{m}$) in the neighborhood of a few kilograms per second, and I_{SP} values in the neighborhood of 100 s for model rocket engines. There seems to be a difference between "choked" and "unchoked" motors, but we won't touch that one either at this level of difficulty.

13-3. *Propulsion: Conservation of Momentum* Imagine that you are sitting in a sled, at rest, on a frictionless surface. You and your sled have mass "*M*". You also have another "fuel" mass, "*m*", with you on the sled. You throw the fuel mass off the back of the sled with speed "*v*" RELATIVE TO YOURSELF. **(a)** Using conservation of momentum, calculate your reaction speed, "V_{11}". **(b)** Now reset the problem (you're at rest again with *m* on your lap) and imagine that you break *m* into two equal pieces, *m*/2. Using conservation of momentum, calculate your reaction speed "V_{12}" after throwing the first half of the fuel off the sled. Then calculate your reaction speed "V_{22}" after throwing the second half of the fuel off the sled (while you're in motion from throwing the first half). [Reminder: you always throw a mass off the sled with speed *v* relative to yourself, regardless of your state of motion.] **(c)** Repeat the problem breaking *m* into three pieces, *m*/3, calculating your speed V_{33}. **(d)** Finally, generalize the problem by breaking the fuel into *N* pieces, *m*/*N*, and calculate your speed, V_{NN}.

SUGGESTED SOLUTION

(a) First, watching from a stationary reference frame (at right), you see me, my sled, and the fuel. Then I throw the fuel off the back of my sled, giving …



$$0 = MV_{11} - mv \quad \Rightarrow \quad V_{11} = \frac{mv}{M}\,.$$

(b) Second, my throwing off half the fuel (top part of sketch below) gives a similar result:

$$0 = \left(M + \frac{m}{2}\right)V_{12} - \frac{m}{2}v \quad \Rightarrow \quad V_{12} = \frac{m/2}{M + m/2}v\,.$$

At this point you can choose to continue to watch me from your stationary frame (at left, below):

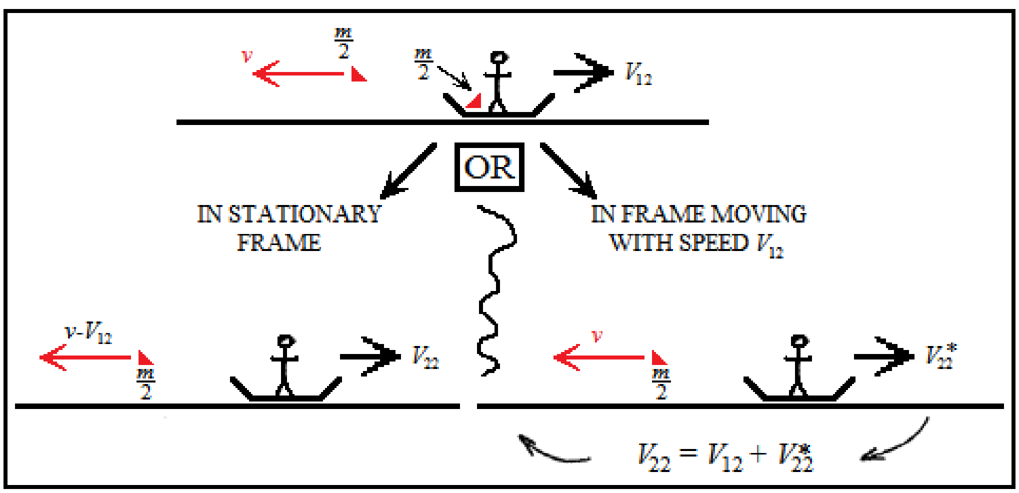
$$\left(M + \frac{m}{2}\right)V_{12} = MV_{22} - \frac{m}{2}\left(v - V_{12}\right),$$

or you can transform to a frame moving with me (at right, below) before I throw the second half:

$$0 = MV_{22}^{*} - \frac{m}{2}v \quad \text{with} \quad V_{22} = V_{12} + V_{22}^{*}\,.$$

Both approaches give the same results (or anything algebraically equivalent) …

$$V_{22} = \frac{mv}{2M}\left[1 + \frac{1}{1 + \frac{1}{2}\frac{m}{M}}\right].$$



(c) Third, if I break my fuel into three pieces, your conservation of momentum calculation (left as an exercise for the student) is

$$V_{33} = \frac{mv}{3M}\left[1 + \frac{1}{1 + \frac{1}{3}\frac{m}{M}} + \frac{1}{1 + \frac{2}{3}\frac{m}{M}}\right].$$

By inspection, you should hopefully start to see a pattern emerging, even though we've only thrown out two pieces of fuel.

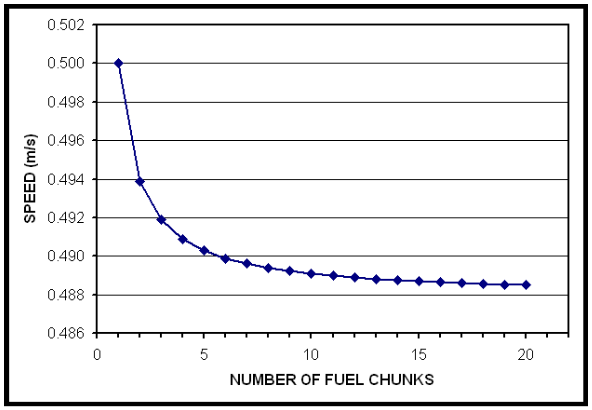
(d) Using the method of reasoning called *induction* (going from specific cases to the most general case), the emerging pattern suggests that when I chop my fuel up into N pieces and throw them off the sled one at a time, I get …

$$V_{NN} = \frac{mv}{NM}\left[1+\frac{1}{1+\frac{1}{N}\frac{m}{M}}+\frac{1}{1+\frac{2}{N}\frac{m}{M}}+\ldots+\frac{1}{1+\frac{N-1}{N}\frac{m}{M}}\right]$$

$$= \frac{mv}{NM}\sum_{i=1}^{N}\left[\frac{1}{1+\frac{i-1}{N}\frac{m}{M}}\right]$$

where the summation notation makes things more compact.

Looking at this sum, "we" wondered what it looked like, so we plotted it (at right). (See the companion spreadsheet for the calculation, dividing the fuel into 1, 2, 3, … 20 chunks, where I chose some values for M, m, and v.) The shape of this curve is not obvious from the generalized summation, but apparently it tells me that I can actually get the most forward speed if I can throw my fuel off the sled in one piece. That's nice, but it may not be practically possible if the value of v (the surrogate for escape gas velocity, v_e) is too high ~ I'm not strong enough to throw the whole thing off the sled at that speed. (See the companion file, "Chapter 13 ~ Suggested Solutions Appendix I" for a complete analysis of the N-chunk problem.)



13-5. *Co-addition of Frames.* In a certain situation we want to locate some nighttime activity that is characterized by some outdoor lighting that amounts to an area with an average in-band radiance of about 10^3 W/m^{2}sr. As a worst case, surrounding events, reflections from external sources, and internal detector and electronic biases and operating conditions provide a random noise floor equivalent to about 5×10^3 W/m^{2}sr in the scene. Clearly, the signal-to-noise ratio (SNR) is prohibitively low to allow reliable detection. However, we know that we can enhance the SNR by the co-addition of frames. If we want to boost our SNR to at least 4 for positive identification, how many frames of data must we co-add?

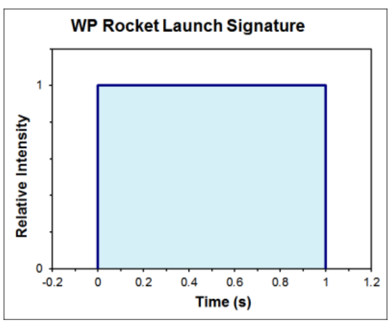
SUGGESTED SOLUTION Astronomers use this all the time. To see a star that is only about 20% as bright as the background noise, they overlay (add with PhotoShop or equivalent) multiple images, say M of them. This artificially enhances the signal-to-noise by a factor $\sqrt{M}$. Here we want to go from 0.2 to 4, so …

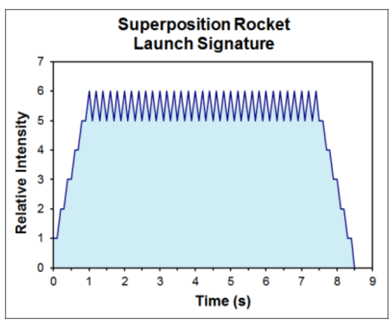
$$4 = \sqrt{M}(0.2) \quad \Rightarrow \quad M = \left(\frac{4}{0.2}\right)^2 = 400.$$

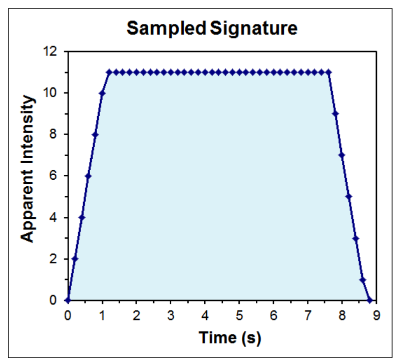
Thus in principle we can enhance the signal to noise by co-adding 400 frames. Other problems associated with this procedure may be (1) do we have the patience to collect 400 frames in the first place, (2) what else might be changing while we're trying to collect the frames (distance to target, background, etc.), and (3) can we co-register the frames all to the same geolocation. There are many other problems associated with this method you could think of.

13-7. ***Temporal Superposition.*** A fighter aircraft carries two pods of 19 each 2.75" white phosphorous (WP, or affectionately "Willy Pete") rockets – one under each wing. If the pilot holds down the trigger and fires them all at a target in about eight seconds (they fire sequentially: left-right-left-right- …), estimate the maximum intensity we would see with an AIRS sensor. Each rocket's launch signature is approximately one second long.

SUGGESTED SOLUTION Below left (next page) is a hypothetical launch signature of a WP rocket as it comes out of its pod ~ modeled as a constant burn for about one second. Supposing the rockets fire at a constant rate from each pod ~ that's 38 rockets per eight seconds, or close enough to approximately 0.20 s per rocket. If the pilot holds the trigger down, their composite launch signature will look something like the middle figure (which is a theoretical superposition of 38 individual signatures at high time resolution ~ see spreadsheet). The maximum apparent intensity is about five or six times that of one rocket, and lasts for a total of about 8.5 seconds. If an overhead sensor, integrating/sampling at 5 Hz say, sees the launch, its output may be somewhat as shown at right. The apparent signature is clearly greater and longer than that of a single rocket. As with other events, it would take more information than just this one collection to ascertain what it was.







13-9. ***Temporal Sampling.*** On the Suggested Problems DATA worksheet for this problem you will find a (synthetic) transient event signature, provided at a time resolution of 0.005 s.

A. Assuming the phase of the event is coincident with the start of a sensor's integration time, show what the collected signature looks like for sampling at 1 Hz, 2 Hz, 5 Hz, and 10 Hz. (Assume 100% duty cycle for simplicity.)

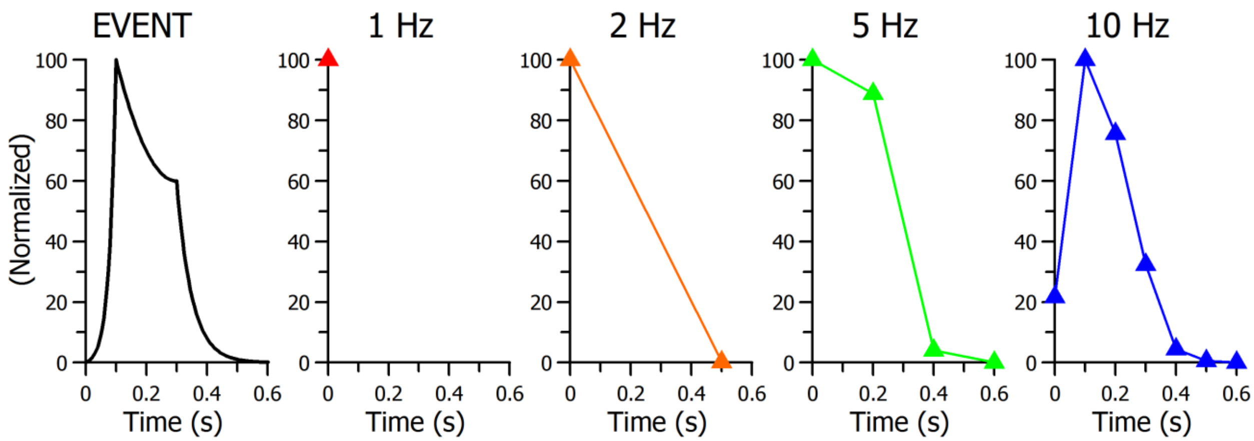
B. Show what the 10 Hz collected signature looks like when the event begins 0.02 s, 0.04 s, 0.06 s, and 0.08 s after the start of the sensor's integration time.

C. Again assuming no phase difference, show what the signature looks like when sampled at 20 Hz, 50 Hz, and 100 Hz.

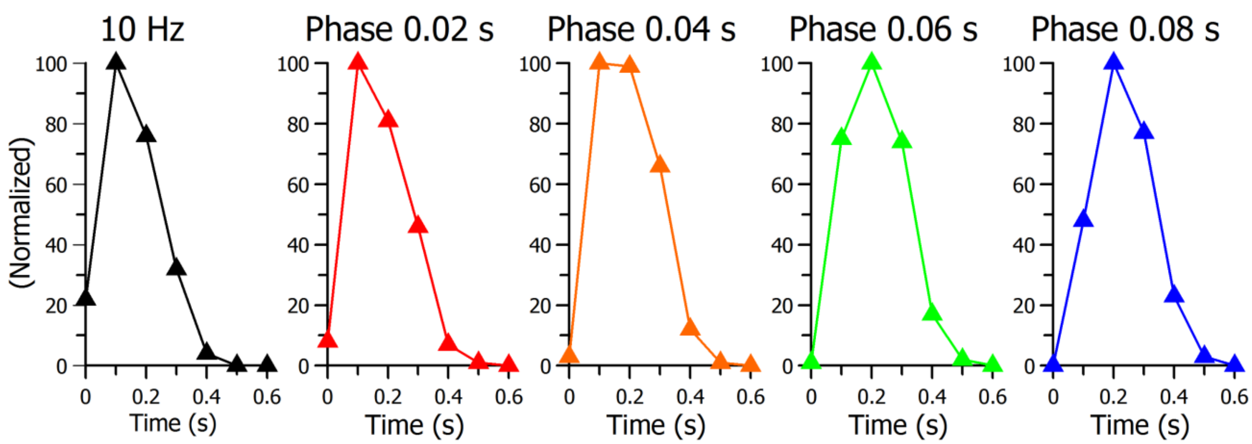
D. What sampling rate is optimum for this event?

SUGGESTED SOLUTION:

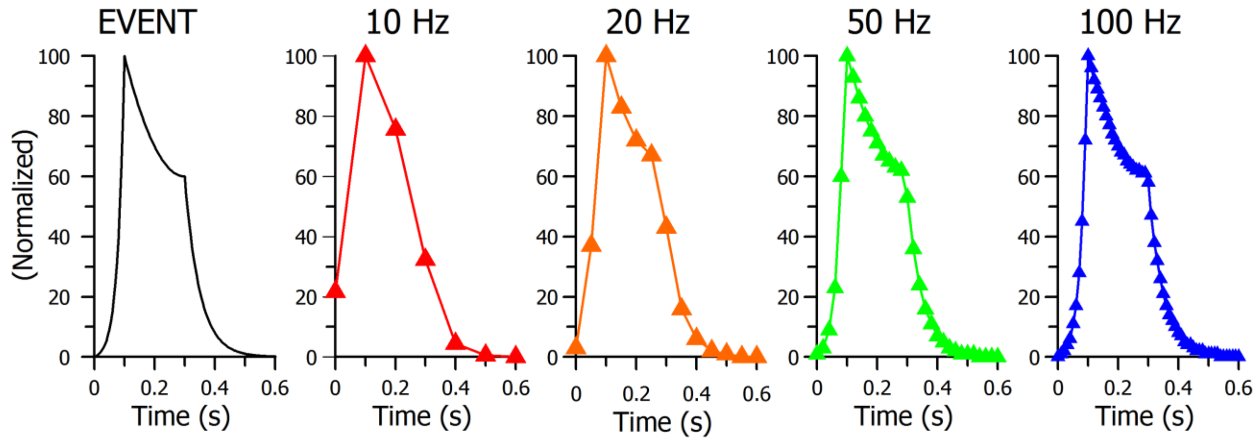
A. The problem does not specify the units for the signature given in the event file, so we will just take it to be something proportional to the amount of energy received by the sensor during each 0.005 second integration period. So instead of integrating the area under the curve – as the sensor temporally integrates its input signal – we will simply add the values given at their collection times. To show the sampled signatures, we will then "normalize" the results (see spreadsheet) to a maximum value of 100 (whatever units) for each signature. We will do the sampling at the various rates by supposing the energy collected during each frame is reported at the beginning of the sampling interval. Here are the results of *undersampling* ~



B. The event signature has been "phased" and sampled on the Suggested Problem Solutions (ODD) worksheet by pushing the given event signature "down" by 0.02, 0.04, 0.06, and 0.08 seconds. The delayed events have then been sampled at 10 Hz as above, and the results are shown here. Note the difference in appearance of the apparent signature.



C. This time the event (zero phase) is *oversampled*. The following plot shows the results.



D. Certainly we see that sampling this transient signature at 1, 2, or 5 Hz is not fast enough to see any details in the changing signature; we would call this undersampling. On the other hand, sampling at 20, 50, or 100 Hz is clearly fast enough to see the details, but these are probably more information than we need, so we would call them oversampled. Furthermore, we must remember the faster we sample the shorter time interval we have to collect energy, so we are going to be more subject to noise (which was not a factor in this simulated signature).

The optimum sampling rate therefore appears to be 10 or 20 Hz to preserve the approximate shape of the signature (at least for this one). To be slightly more analytical, we note that the fastest rate of change of the signature occurs between 0.0 and 0.1 seconds. To adequately sample this change, according to the Nyquist sampling theorem, we should therefore sample at about every 0.05 seconds, or 20 Hz. This is satisfactory for this particular (synthetic) signature, but is not a general rule, although the difficulties associated with faster sampling – low integrated energy and too much data to process – may limit practical sensors to about this maximum rate.