SUGGESTED SOLUTIONS (ODD)

CHAPTER 7

NOTE: Use three-digit precision for all calculations unless otherwise stated or implied.

7-1. *Work Function.* Is the energy of a $\lambda = 1.2\,\mu\text{m}$ photon sufficient to generate a photoelectron from a material having a work function – or band gap energy (between valance and conduction bands) – of $\phi = 1.2\,\text{eV}$?

SUGGESTED SOLUTION: The energy of a $\lambda = 1.2\,\mu\text{m}$ photon is approximately

$$E_{PHOTON} = \frac{hc}{\lambda} = \frac{1.24}{\lambda(\mu m)} \approx 1.03\,\text{eV}$$

Since $E_{PHOTON} < \phi$, this photon does not have sufficient energy to promote an electron across the band gap between the valance and conduction bands. So the short answer is "NO."

7-3. *Photodetector Output.* A photodetector receives $\text{E} = 1.09\times10^{-6}\,\text{W/m}^2$ of $\lambda = 2.67\,\mu\text{m}$ light on its surface from an external source. Its area is $A_d = 100\times100\,\mu\text{m}$, its quantum efficiency is $\eta = 0.35$, and its integration time is $\Delta t_{INT} = 0.10\,\text{s}$. What is the number of electrons, $\tilde{N}$, output by this detector in one sample?

SUGGESTED SOLUTION: Phenomenologically in words ~

$$\frac{\text{Electrons}}{\text{Sample}} = \frac{\text{Power}}{\text{Area}} \times \text{Area} \div \frac{\text{Energy}}{\text{Photon}} \times \frac{\text{Electrons}}{\text{Photon}} \times \frac{\text{Time}}{\text{Sample}}$$

which translates into symbols ~

$$\tilde{N} = \text{E} \times A_d \times \frac{1}{hc/\lambda} \times \eta \times \Delta t_{INT} = \frac{\text{E}\,A_d\,\lambda\,\eta\,\Delta t_{INT}}{hc}$$

and then into numbers ~

$$\tilde{N} = \frac{\left(1.09\times10^{-6}\,\text{W/m}^2\right)\left(10^{-4}\,\text{m}\right)^2\left(2.67\times10^{-6}\,\text{m}\right)(0.35)(0.10\,\text{s})}{\left(6.63\times10^{-34}\,\text{J}\cdot\text{s}\right)\left(3\times10^{8}\,\text{m/s}\right)} \approx 5120\text{ electrons}$$

7-5. *Digitized Output.* If the detector in Problem 7-3 is designed to saturate for a source that is 20 times brighter than the source detected in that problem, and its output is digitized to 12 bits, what is its digitized output when it sees a source giving an irradiance of $\text{E} = 5.7\times10^{-6}\,\text{W/m}^2$ on its surface?

SUGGESTED SOLUTION: See the suggested solution for Problem 7-7.

7-7. ***More Digitized Output.*** If the detector in Problem 7-3 is designed to saturate for a source that is 20 times brighter than the source detected in that problem, and its output is digitized to 12 bits, what is the irradiance on its surface if its output is 1945 Digital Units (DU)?

SUGGESTED SOLUTION (for Problems 7-5 and 7-7): For both problems, we calculate that the "saturation" input to our photodetector is $E_{SAT} = 20 \times 1.09 \times 10^{-6}\ \text{W/m}^2 \approx 2.18 \times 10^{-5}\ \text{W/m}^2$. Using the same proportionality (or the phenomenological equation we developed for Problem 7-3), the saturated output is then $\tilde{N}_{SAT} = 20 \times 5120\ \text{electrons} = 102,400\ \text{electrons}\ (\text{e}^-)$. This, we are told, would be scaled to $2^{12} = 4096$ divisions, or bins, on a new scale (0 through 4095 inclusive) which can be represented by a 12-bit binary number. (The divisions are called digital units (DU) which are visually shown as shades of gray.) Accordingly, each bin (i.e., "bit") represents an output of 102,400 e^- ÷ 4096 bits ≈ 25 e^-/bit. Hence, for example, our digitized detector output for Problem 7-3 is 5120 ÷ 25 = 204.8 bits ⇒ 205th bin ⇒ 204 DU (because the first bin is number zero).[1] To convert this to binary, we note that 204 = 128 + 64 + 8 + 4 = $2^7 + 2^6 + 2^3 + 2^2$. Using 1's and 0's, this number[2] is 000011001100 in binary. These results are shown in the table below.

For Problem 7-5, we find, proportionately, that the given input represents

$$\frac{5.70 \times 10^{-6}\ \text{W/m}^2}{2.18 \times 10^{-5}\ \text{W/m}^2} \times 102,400\ \text{e}^- \approx 26,780\ \text{e}^-, \text{ which is } \frac{26,780\,\text{e}^-}{25\,\text{e}^-/\text{bit}} = 1071.2 \Rightarrow 1072\ \text{bits} \Rightarrow$$

$1071\ \text{DU}$. These results are also shown in the table, together with the binary representation of 1071 = 1024 + 32 + 8 + 4 + 2 + 1 = $2^{10} + 2^5 + 2^3 + 2^2 + 2^1 + 2^0$.

For Problem 7-7, the meaning of the given 1945 DU is that we have electrons filling at least 1945 bins (1945 × 25 = 48,625 e^-), but possibly with some electrons in the 1946th bin (48,650 e^-). [COMMENT: at the level of precision we are used to dealing with – three significant digits usually, although we slip in a fourth once in a while when it's convenient – it turns out that we don't have to worry about one or two DUs here and there; we can probably tolerate a 100 electron (4 DU) ambiguity.] Again using proportionality, we calculate that the input to the photodetector that gave us this output was ~

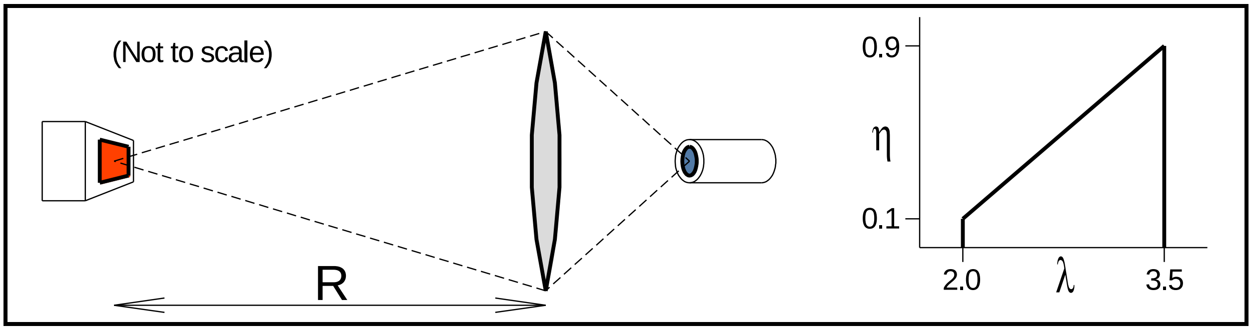
$$\frac{48,625\ \text{e}^-}{102,400\ \text{e}^-} \times 2.18 \times 10^{-5}\ \text{W/m}^2 \approx 1.04 \times 10^{-5}\ \text{W/m}^2$$

This result is shown in the third line of the following table. The table summarizes our calculations for three problems, with givens shown in blue and results in red.

[1] Note there is a "quantum ambiguity." Any number of electrons between 5101 and 5125 would give us the same bin number, or DU. Thus we cannot go backwards with certainty from DU to number of electrons (Problem 7-7).

[2] There are only 10 kinds of people: those who can read binary and those who can't.

	Irradiance on photodetector (W/m^2)	Photodetector output (electrons)	Photodetector output (DU)	Photodetector output (binary)
Problem 7-3	1.09×10^{-6}	5120	204	000011001100
Problem 7-5	5.70×10^{-6}	26,780	1071	010000101111
Problem 7-7	1.04×10^{-5}	48,625 to 48,650	1945	011110011001
"Saturation"	2.18×10^{-5}	102,400	4095	111111111111



7-9. *Calculating Detector Output.* In a laboratory, radiation from a one-inch square, T = 350 K diffuse blackbody source is collected by a D_L = 4" diameter f/5 lens R = 2.0 m away and is focused on a Hg:Cd:Te photodetector with the quantum efficiency shown above. The detector has a D_D = 1" diameter photosensitive surface. The detector is operated at f_D = 10 Hz with a $\overline{DC} = 99\%$ duty cycle. What is the output of the detector in electrons per integration time, $\tilde{N}$?

SUGGESTED SOLUTION: Let's look at a phenomenological solution:

First, the source is a Lambertian blackbody, emitting radiation $B_\lambda(\lambda,T)$ [W/(m^{2}μm)] which is its *spectral* exitance. The *spectral* radiance is therefore $L_\lambda = \dfrac{B_\lambda}{\pi}$ [W/(m^{2}sr μm)]. From the lens, the source subtends a solid angle $\Omega \approx \dfrac{X\,Y}{R^2}$ [sr] where X and Y are the linear dimensions of the source (X = Y = 1 inch = 0.0254 m). Since the source dimensions are small compared to R (2 m), we can approximate the *spectral* irradiance on the lens from the source as $E_\lambda \approx L_\lambda \Omega = \dfrac{B_\lambda}{\pi}\dfrac{X\,Y}{R^2} = \dfrac{X\,Y\,B_\lambda}{\pi\,R^2}$ [W/(m^{2}μm)]. The *spectral* power passing through the lens, and imaged onto the detector's surface is thus $\Phi_\lambda = E_\lambda A_R = \dfrac{X\,Y\,B_\lambda}{\pi\,R^2}\dfrac{\pi\,D_L^2}{4} = \dfrac{X\,Y\,D_L^2 B_\lambda}{4\,R^2}$ [W/μm]

where D_L is the lens diameter ($D_L = 4\text{ in} = 0.1016\text{ m}$). Note that we are assuming both the atmospheric and optical transmission functions (τ_{ATM} and τ_{OPT}) to be unity. Note also – and this is really important – that we have carried the "per unit bandpass" or "per micron" along with us through the calculation so far. This is because we haven't applied a bandpass yet.

Dropping back a moment, note that the aperture stop ($f/\# = \frac{f_{eff}}{D_L}$, where f_{eff} is the focal length and D_L is the aperture diameter) implies that the focal length of the collecting lens is $f_{eff} = (f/\#) \times D_L = 5 \times 1.1016\,\mathrm{m} = 0.508\,\mathrm{m}$. Using the image finder's formula, $\frac{1}{d_O} + \frac{1}{d_I} = \frac{1}{f_{eff}}$, where d_O is the object distance (R in our case) and d_I is the image distance, we calculate the image distance to be $d_I = \frac{d_O f_{eff}}{d_O - f_{eff}} = \frac{2\,\mathrm{m} \times 0.508\,\mathrm{m}}{2\,\mathrm{m} - 0.508\,\mathrm{m}} = 0.681\,\mathrm{m}$. We then have that the image magnification is $|M| = \left|\frac{d_I}{d_O}\right| = \frac{0.681}{2} = 0.340$, and hence the size of the image of the blackbody source on our photodetector is approximately 2.54 cm × 0.340 = 0.865 cm on a side. This is sufficiently small that we can assume that all of the photons we will calculate next will fall on the detector's surface and be converted into electrons through its quantum efficiency.

Now, the energy per photon (wavelength in meters) arriving at the detector is $E = \frac{h\,c}{\lambda}$ [J] so the *spectral* number of photons per second on the detector is $\dot{P}_\lambda = \frac{\Phi_\lambda}{E} = \frac{X\,Y\,D_L^2 B_\lambda \big/ 4\,R^2}{h\,c \big/ \lambda}$

$= \frac{X\,Y\,D_L^2 \lambda\, B_\lambda}{4\,h\,c\,R^2}$ [photons/(s-μm)] (note that the dot over the P is Newton's "per unit time").

Since the integration time is $\Delta t_{INT} = \overline{DC} \times \frac{1}{f_D}$ [s] where $\overline{DC}$ is the "duty cycle" ($\overline{DC}$ = 99% = 0.99) and f_D is the operating frequency (f_D = 10 Hz), the number of photons arriving at our detector per integration time is $P_\lambda = \dot{P}_\lambda \Delta t_{INT} = \frac{X\,Y\,D_L^2 \overline{DC}\,\lambda\, B_\lambda}{4\,h\,c\;f_D\;R^2}$ [photons/μm]. Note that we still have a *spectral* quantity, and photons of different wavelengths get converted into electrons by the detector's quantum efficiency, which we will discuss next.

Between the wavelengths of 2.0 and 3.5 micrometers, our HG:Cd:Te photodetector has a quantum efficiency that increases linearly ($\eta = m\lambda + b$) according to a relation that we can derive[3] from the chart: $\eta = \frac{0.8}{1.5}\lambda - \frac{1.45}{1.5}$ [electrons/photon] where the first constant (slope) has units of electrons/(photon-μm) and the second constant (intercept) has units of electrons/photon. Evidently the number of electrons generated in one integration time is then the product:

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[3] You can read the slope off the chart directly: $\Delta y = \Delta\eta = 0.9 - 0.1 = 0.8$, and $\Delta x = \Delta\lambda = 3.5 - 2.0 = 1.5$; thus $m = \frac{\Delta y}{\Delta x} = \frac{\Delta\eta}{\Delta\lambda} = \frac{0.8}{1.5}$. Finding the intercept, b, requires plugging in a known point on the line and solving. For example, when (λ, η) = (2.0, 0.1) we have $0.1 = \frac{0.8}{1.5}2.0 + b$, and solving gives us $b = 0.1 - \frac{(0.8)(2.0)}{1.5} = -\frac{1.45}{1.5}$.

$\tilde{N}_\lambda = P_\lambda \eta = \dfrac{X\,Y\,D_L^2\,\overline{DC}\,\lambda\,B_\lambda}{4\,h\,c\,f_D\,R^2}\left(\dfrac{0.8}{1.5}\lambda - \dfrac{1.45}{1.5}\right)$ [electrons/μm]. Now if you are sharp-eyed, you will notice two peculiar things about this expression. FIRST, the first λ in the formula is in units of meters – having come from our calculation of photon energy – while the second λ is in units of microns – from the quantum efficiency relation. We will fix this below. SECOND, we still have a *spectral* quantity. To derive our final answer, we have to integrate this over the sensor's bandpass: $\tilde{N} = \int\limits_{BANDPASS} \tilde{N}_\lambda d\lambda$

Apparently, we will have to be extra careful of the units in our calculation of the integral. We note that the Planck formula, $B_\lambda = \dfrac{c_1}{\lambda^5\left(\exp\left(c_2/\lambda T\right)-1\right)}$, calls for wavelength in units of micrometers when we use the values $c_1 = 3.742\times10^8$ [W-μm^4/m^2] and $c_2 = 1.438\times10^4$ [μm-K] for the so-called "first and second radiation constants" as given in our text. It only makes sense, therefore, that we should be consistent and use micrometers for all instances of wavelength in our calculation. When we do this, let's see what happens.

Putting everything together, we see that we want to integrate:

$$\begin{aligned}\tilde{N} &= \int\limits_{BANDPASS} \frac{X\,Y\,D_L^2\,\overline{DC}}{4\,h\,c\,f_D\,R^2}\,\lambda\,B_\lambda\left(m\lambda+b\right)d\lambda \\ &= \frac{X\,Y\,D_L^2\,\overline{DC}}{4\,h\,c\,f_D\,R^2}\int\limits_{BANDPASS}\lambda\,B_\lambda\left(m\lambda+b\right)d\lambda\end{aligned}$$

Since we want to do the integral over wavelength in units of microns, a check on the units[4,5]:

$$\frac{[\mathrm{m}][\mathrm{m}]\left[\mathrm{m}^2\right]}{[\mathrm{J\cdot s}]\left[\frac{\mathrm{m}}{\mathrm{s}}\right]\left[\frac{1}{\mathrm{s}}\right]\left[\mathrm{m}^2\right]}\int[\mu\mathrm{m}]\left[\frac{\mathrm{W}}{\mathrm{m}^2\,\mu\mathrm{m}}\right][\mu\mathrm{m}] = \left[\frac{\mu\mathrm{m}}{\mathrm{m}}\right]$$

shows that we have some dimensions left over – remember our final answer is just supposed to be in "electrons" (per integration period) which is just a number with no units. Thus to make things work out right, we have to stick in a conversion factor, $K = \dfrac{10^{-6}\,\mathrm{m}}{1\,\mu\mathrm{m}}$. With this factor in place, the coefficient of fixed quantities out in front of the integral evaluates to

$$\frac{X\,Y\,D_L^2\,\overline{DC}\,K}{4\,h\,c\,f_D\,R^2} = \frac{(0.0254\,\mathrm{m})(0.0254\,\mathrm{m})(0.1016\,\mathrm{m})^2(0.99)(10^{-6}\mathrm{m}/\mu\mathrm{m})}{(4)(6.626\times10^{-34}\,\mathrm{J\cdot s})(2.998\times10^{8}\mathrm{m/s})(10\,\mathrm{Hz})(2\,\mathrm{m})^2} = 2.07\times10^{11}\,\frac{\mathrm{m}^2}{\mathrm{W}\cdot\mu\mathrm{m}},$$

(assuming three significant digits throughout), and the integral apparently has units of $\dfrac{\mathrm{W}\cdot\mu\mathrm{m}}{\mathrm{m}^2}$.

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[4] The notation has shown here only sticks in the units for those terms in the formula that have them.

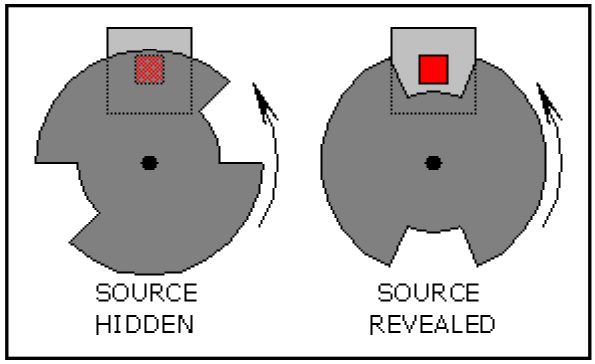
[5] Remember that the "$d\lambda$" also has units of micrometers!

The integral is done numerically in the companion spreadsheet where we have used simple trapezoidal integration to derive its value 5.22. So, finally, we have our estimate that ~

$$\left(2.07\times10^{11}\frac{\mathrm{m}^2}{\mathrm{W}\cdot\mu\mathrm{m}}\right)\left(5.22\frac{\mathrm{W}\cdot\mu\mathrm{m}}{\mathrm{m}^2}\right)=1.08\times10^{12}\,\text{electrons}$$

are generated (per integration time).

7-11. ***An Experimental Photodetector Test.*** The laboratory set-up in the previous problems (7-9 and 7-10) is modified to include a "chopper wheel" in front of the blackbody source. The wheel is eight inches in diameter, has two 45° wide openings, and spins in front of the blackbody – alternately covering and exposing it to the photodetector. First calculate the photodetector's output when the blackbody is hidden and then when it is revealed. Determine the maximum output of the detector when the wheel spins at rates of 2, 5, 10, and 20 Hz (i.e., revolutions per second). (Recall that the detector integrates at 10 Hz with a 99% duty cycle.)



SUGGESTED SOLUTION: First, we are going to modify our calculations from Problems 7-9 and 7-10 to compute the *rate* at which our photodetector is converting incident photons into output electrons. You will understand why later.

From Problem 7-9, our photodetector's output when seeing only the blackbody is ~

$$\dot{N}_{BB}=\int_{BANDPASS}\frac{X\,Y\,D^2}{4\,h\,c\,R^2}\lambda\,B_\lambda\left(m\lambda+b\right)d\lambda=\frac{X\,Y\,D^2K}{4\,h\,c\,R^2}\int_{2.0\mu\mathrm{m}}^{3.5\mu\mathrm{m}}\lambda\,B_\lambda\left(m\lambda+b\right)d\lambda$$

$$=\left(2.095\times10^{12}\frac{\mathrm{m}^2}{\mathrm{W}\cdot\mathrm{s}\cdot\mu\mathrm{m}}\right)\left(5.22\frac{\mathrm{W}\cdot\mu\mathrm{m}}{\mathrm{m}^2}\right)\approx1.09\times10^{13}\mathrm{s}^{-1}$$

And from Problem 7-10, our photodetector's output when seeing the background[6] surrounding the blackbody is ~

$$\dot{N}_{BG}=\int_{BANDPASS}\left(\frac{\pi x^2}{4d_{IMG}^2}-\frac{XY}{R^2}\right)\frac{D^2}{4hc}\lambda B_\lambda\left(m\lambda+b\right)d\lambda=\left(\frac{\pi x^2}{4d_{IMG}^2}-\frac{XY}{R^2}\right)\frac{D^2K}{4hc}\int_{2.0\,\mu m}^{3.5\,\mu m}\lambda B_\lambda\left(m\lambda+b\right)d\lambda$$

$$=\left(1.21\times10^{13}\frac{\mathrm{m}^2}{\mathrm{W}\cdot\mathrm{s}\cdot\mu\mathrm{m}}\right)\left(0.636\frac{\mathrm{W}\cdot\mu\mathrm{m}}{\mathrm{m}^2}\right)\approx7.69\times10^{12}\mathrm{s}^{-1}$$

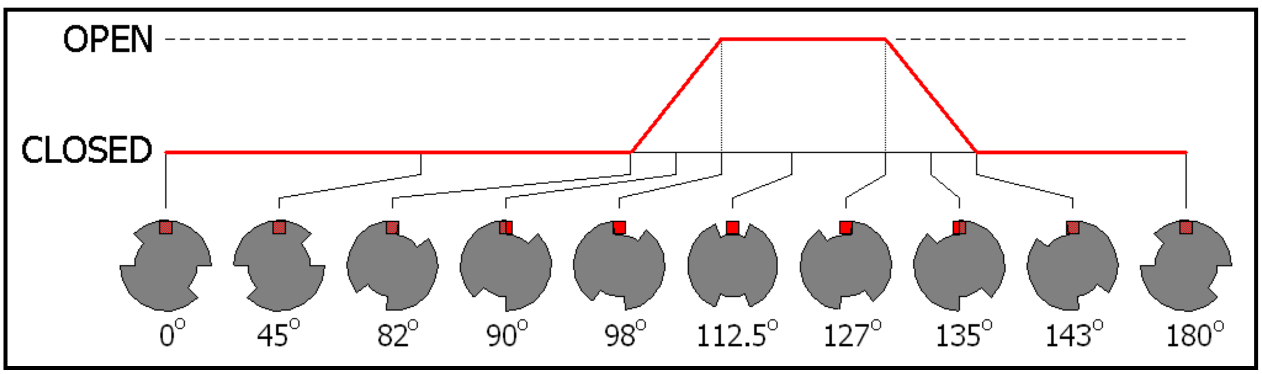
Combining these results, we have that our photodetector's output when seeing the blackbody source surrounded by its background is 1.86×10^{13} electrons per second.

[6] We are assuming the chopper wheel is at 300 K, the same temperature as the background in Problem 7-10, so when the photodetector sees partly laboratory space and partly chopper wheel surface, it is all the same.

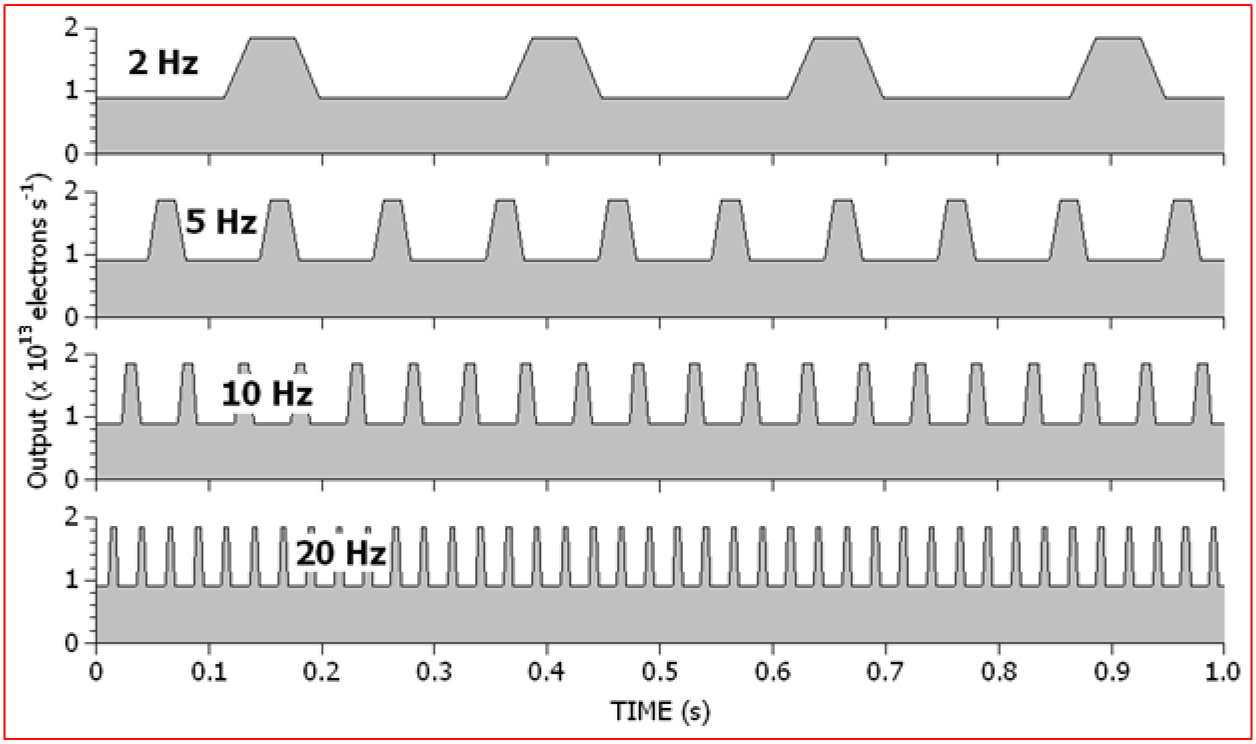
Second, we are going to add to these results a third calculation for our photodetector's output when the blackbody source is concealed by the chopper wheel. We will assume then that it sees only a 300 K background (chopper wheel plus whatever other apparatus may be behind it). The only difference from the previous problems is the solid angle filled with extended radiant background is now $\Omega_{BG} = \frac{\pi x^2}{4d_{IMG}^2}$. Using this FOV, the student can easily verify that our photodetector's output at these times is ~

$$\dot{N}_{BG} = \frac{\pi x^2 D^2 K}{16hcd_{IMG}^2} \int_{2.0\mu\text{m}}^{3.5\mu\text{m}} \lambda B_\lambda (m\lambda - b) d\lambda$$
$$= \left(1.42\times10^{13} \frac{\text{m}^2}{\text{W}\cdot\text{s}\cdot\mu\text{m}}\right)\left(0.636\frac{\text{W}\cdot\mu\text{m}}{\text{m}^2}\right) \approx 0.903\times10^{13}\text{s}^{-1}$$

Third, we will address the issue of the chopper wheel. The sketch below shows the progression of one its slots across the blackbody through half a revolution. As you see, the photodetector sees at least some of the blackbody for (143° – 82°)/180° ≈ 33.9% of the time while it is fully uncovered for only about (127° – 98°)/180° ≈ 16.1% of a rotation. The transition from covered to uncovered is, technically, a convolution of the two shapes (slot in wheel and blackbody surface) passing over one another, but we will just approximate it with a linear rise and fall.



Since our photodetector's output is known when it is seeing the blackbody – and not – we can plot its output (electrons per second) as a function of time when the chopper wheel is rotating at speeds of 2, 5, 10, and 20 Hz, thusly ~



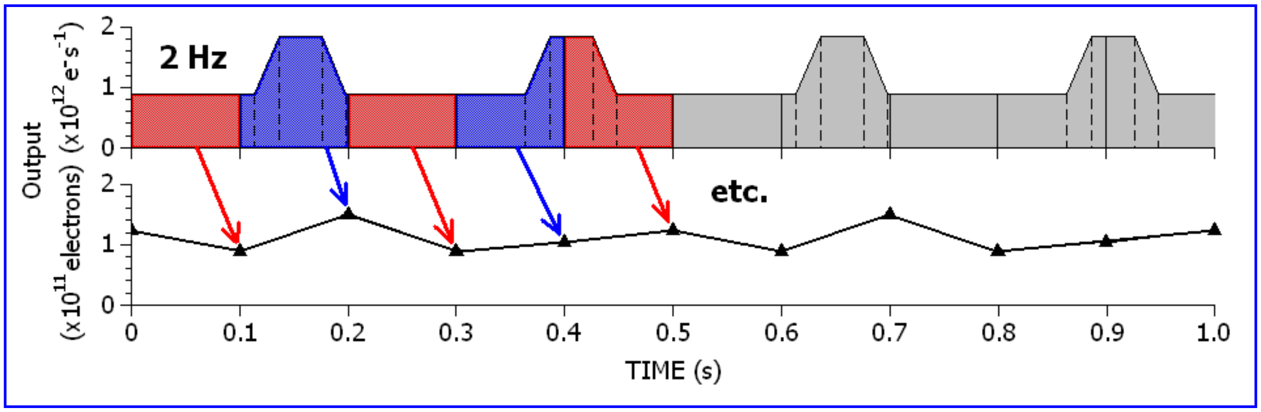
BUT WAIT, there's more because the photodetector integrates its input …

With our photodetector operating in an integrating-sampling mode, its output is ~

$$N = \int \dot{N}(t)\,dt \quad [\text{electrons}]$$

which is usually interpreted as being the area under the curve. For the remainder of this problem, let us assume that the rotation of the chopper wheel is synched to the integration cycle of our photodetector, such that angle zero in the figure above corresponds to the beginning of an integration interval. (This need not necessarily be so, and it is a very important issue which should be explored in the interpretation of data from non-literal OPIR sensors in Chapters 13 and 14.) Furthermore, with a duty cycle of 99%, we will ignore the readout time.

For the chopper wheel rotating at 2 Hz, the output of our photodetector (in electrons per integration time) is as shown below. Because of our simplifying assumptions, the calculation of the temporal integral is just reduced to the areas of rectangles and trapezoids. If we let the photodetector's output be reported at the end of each of its samples, and fancifully "connect the dots," we see that its output only mildly resembles the input. Is this sufficient for us to be able to detect the presence of the blackbody in the photodetector's FOV? This is a matter for advanced study in signal processing, but we could imagine that – presented with these data – we could probably say that there is something there besides background, but, obviously, we are unable to identify it. We have nibbled at the edge of non-literal signature identification.



When the rotation speed on the chopper wheel increases, the situation becomes as shown in the next figure. The photodetector's temporal integration is no longer able to discern that there is any target present in its FOV at all! Again, we have stepped into an area where we need to rethink how we want to collect and process these data.