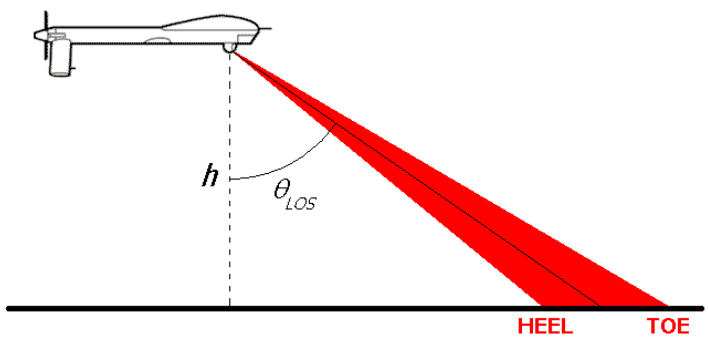
SUGGESTED SOLUTIONS (ODD)

CHAPTER 12

NOTE: Use three-digit precision for all calculations unless otherwise stated or implied. You will discover that more precision is definitely needed for a couple of these problems.

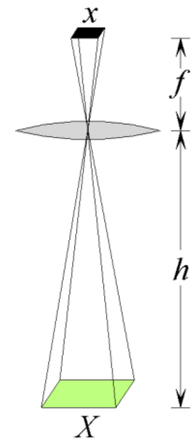
12-1. *Unmanned Aerial Vehicle.* Suppose a UAV could fly at 20,000 ft. above the terrain at 250 kts. In its nose sensor pod it carries an IR camera with a fixed field of view 3° high × 10° wide, having its boresight fixed and pointing forward at a 50° nadir angle. Borrowing from conventional technology, the sensor frames at a video rate of $f \approx 24$ Hz. If a bright point source dead ahead of the aircraft comes into the field of view, how many full frames of data can be taken of it?



SUGGESTED SOLUTION: Nothing too tricky with this problem. First, we note that the platform's altitude is $h \approx 20{,}000$ ft ≈ 6096 m, and its speed is $v \approx 250$ kt ≈ 128.6 m/s. With a boresight nadir angle of $\theta_{LOS} \approx 50°$ and a vertical FOV angle of 3º, the sighting angle to the "heel" of the sensor's footprint (see drawing at left) is 48.5º and the "toe" angle is 51.5º. Then we can compute the distance from the UAV's sub-point to the heel of its IFOV as $d_{HEEL} \approx h \times \tan(\theta_{HEEL}) \approx 6890\text{ m}$ and similarly the toe distance is ≈ 7630 m. The fore-aft length of the footprint is therefore ≈ 740 m. At its present speed, it will take the predator $\Delta t \approx {}^{740\text{ m}}\!/\!_{128.6\text{ m}\cdot\text{s}^{-1}} \approx 5.75\text{ s}$ to fly this far. At 24 Hz, the number of frames we can expect to take of a point source is thus $\# \approx 5.75\text{ s} \times 24\text{ Hz} \approx 138\text{ frames}$.

This answer is assuming the point source comes into the field of view exactly as one frame is starting, but this is not to be expected in general. More likely, the target will come into the FOV during one frame (frame time is ${}^{1}\!/\!_{24\text{ Hz}} \approx 0.0417\text{ s}$, during which the sensor moves $d \approx {}^{128.6\text{ m}\cdot\text{s}^{-1}}\!/\!_{24\text{ Hz}} \approx 5.36\text{ m}$). In this case, the sensor will only be able to collect 137 full frames of data.

12-3. ***Low Altitude Imaging.*** A low-altitude satellite (400 km near-circular orbit) carries a sensor payload with a fixed, nadir-pointing 10° × 10° FOV. If the sensor's $D \approx 25$ cm aperture optical system is reasonably "fast" ($f/\# \approx 4.0$), how many pixels does it require on its focal plane to achieve (a) a 50 m GSD? (b) a 5 m GSD? and (c) a 50 cm GSD? What is the sensor's spatial resolution in each of these cases? What is this sensor's most likely bandpass?



SUGGESTED SOLUTION: With a 10° nadir pointing FOV from 400 km, we can easily calculate that the width of this sensor's IFOV is approximately $X \approx h\theta = (400\text{ km})(10°)\left(\dfrac{\pi}{180°}\right) \approx 69.813\text{ km}$ wide. For 50 m, 5 m, and 0.5 m GSD, we will need[1]

$$\frac{69.813\text{ km}}{0.05\text{km, }0.005\text{km, }0.0005\text{km}} \approx 1400, 14,000, \text{ and } 140,000 \text{ pixels}.$$

That is, we need FPAs that are about 2 mega-, 200 mega-, and 20 gigapixels.

At first blush, it would seem the first one is fairly easy to build as a single chip with today's technology, but the larger ones may have to be manufactured in pieces and assembled. However, let's look at another aspect of this focal plane. From the given f-stop, we calculate that the system focal length is about $f = (f/\#)D \approx (4.0)(25\text{ cm}) \approx 100\text{ cm}$. By proportional triangles, this gives the dimension of the focal plane to be $x \approx f\theta \approx (100\text{ cm})(10°)\left(\dfrac{\pi}{180°}\right) \approx 17.5\text{ cm}$. For the number of pixels we need, their pitch must be $\dfrac{17.5\text{ cm}}{1400, 14,000, 140,000 \text{ pixels}} \approx 125\mu\text{m}, 12.5\ \mu\text{m, and } 1.25\ \mu\text{m}$. Considering the size and pitch, the middle value is probably the only reasonable choice, within today's technology. We will therefore limit our further discussion to only this proposed FPA.

As for spatial resolution, the answer depends on the purpose of this sensor: for literal imagery, it would seem to be 5 meters (one GSD separation), but for non-literal imagerythe answer looks like perhaps 2 × 5 meters ≈ 10 meters. However, we need to examine the proposed pixel size compared to the point spread function, which depends on the wavelength.

Writing the radius of the Airy disk as $r_A = 1.22\dfrac{f}{D}\lambda = 1.22\left(f/\#\right)\lambda$, we can construct the following table for the possible range of electro-optical wavelengths:

[1] The notation in this equation is a little non-standard, but should be obvious.

λ	r_A	#Pixels	#Pixels
(μm)	(μm)	(aligned)	(diagonal)
0.4	1.95	0.16	0.11
0.7	3.42	0.27	0.19
1.1	5.37	0.43	0.30
2.0	9.8	0.78	0.55
3.5	17.1	1.37	0.97
5.0	24.4	1.95	1.38
8	39	3.12	2.21
10	49	3.90	2.76
12	59	4.68	3.31

The third and fourth columns of the table give the size of the PSF in number of pixels (aligned with the rows/columns of the FPA, and worst case diagonally, respectively). We see that any wavelength up to about 2.0 μm would make this sensor useful only for imagery type imaging; the Nyquist criterion is not satisfied until something beyond 5.0 μm. The thermal band (8 – 12 μm), on the other hand, is good only for non-literal imaging. Together with the information that this is a fairly low-altitude sensor, our guess is that its purpose in life is to take literal images; thus we infer that its most likely bandpass is the visible through shortwave infrared.

12-5. ***A Staring and Scanning Sensor.*** Consider an experimental sensor having a square, 256 (16 × 16) pixel focal plane array detector having a $\theta_x \times \theta_y \approx 16° \times 16°$ field of view. The boresight of the sensor's optics is perpendicular to the velocity vector of its host platform, establishing the *x*-axis of its FOV as being along-track and its *y*-axis as being cross-track. The satellite is in a perfectly circular, $z \approx 300\ \text{km}$ altitude orbit (around a perfectly spherical Earth). The sensor is operated at a 10% duty cycle with an integration time $\Delta t_{INT} \approx 0.1\,\text{s}$.

A. Suppose that the boresight is locked, pointing straight down at the satellite's nadir. (The sensor's Pointing-Tracking-Stabilization (PTS) system eliminates roll/yaw errors, but the pitch continuously changes at the exact rate of the sensor's revolution around the Earth to maintain nadir pointing.) What is the sensor's IFOV and GSD? [Assume a flat Earth.]

B. If the satellite passes directly over a stationary point source having an intensity I_0, what is the sensor's output as a function of time?

C. Now suppose the sensor (more correctly, a rotating scanning mirror) is spun up to a rate of $f_{ROT} = 1.0\ \text{Hz}$ about an axis along its velocity vector. Calculate the sensor's output as a function of time as it passes over the point source.

SUGGESTED SOLUTION:

A. For a first cut, the sensor's IFOV and GSD, using the small angle, flat Earth approximation, are

$$IFOV \approx z\,\theta \approx (300\text{ km})\left(16^\circ \times \frac{\pi\text{ rad}}{180^\circ}\right) \approx 83.776\text{ km}$$

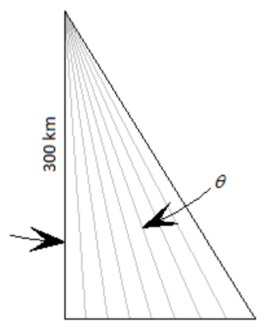
$$\text{and } GSD \approx \frac{IFOV}{16} \approx \frac{83.78\text{ km}}{16} \approx 5.236\text{ km},$$

where the calculated dimensions are the same in both the *x*- and *y*-directions owing to the FOV's symmetry. A slightly more accurate (still flat Earth) answer, however, is

$$IFOV \approx 2z\tan\frac{\theta}{2} \approx (2)(300\text{ km})\tan\frac{16^\circ}{2} \approx 84.32\text{ km}$$

$$\text{and } GSD \approx \frac{IFOV}{16} \approx \frac{84.32\text{ km}}{16} \approx 5.270\text{ km}.$$

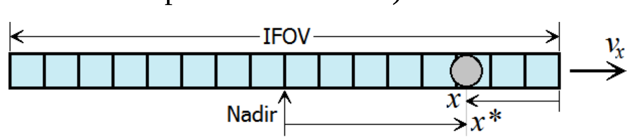
To be picky, however, we note that the pixel footprints (GSDs) are slightly larger at the edges of the IFOV than at the center. The exaggerated sketch at right shows this (showing $\theta = 5^\circ$ off nadir), and the table calculates the GSDs, showing the outermost rows/columns are about 90 m larger than the middle ones.



θ (deg)	From Nadir (km)	GSD (km)
0	0.000	5.237
1	5.237	5.240
2	10.476	5.246
3	15.722	5.256
4	20.978	5.269
5	26.247	5.285
6	31.531	5.304
7	36.835	5.327
8	42.162	

Ultra-picky students will want to refine this calculation even more by taking into account the curvature of the Earth. Since we did this once in Problem 12-4, we won't do it again here. To make life simpler, however, we will proceed with our first approximation (the problem will get complicated enough without anything else).

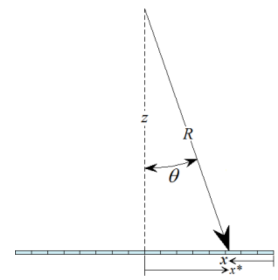
B. This part can be made as difficult as you want, so let's start off with some assumptions. First, since we don't know the bandpass, and we're only given a constant value for target intensity, we'll assume the bandpass is "narrow," so wavelength dependence has been taken care of and the integral is done. Second, since we don't know the focal length or aperture, we'll assume this low altitude sensor is an experiment looking for a source on/source off condition ~ the PSF is approximately the size of one pixel, and we will suppose that it tracks along one column of pixels as the sensor passes overhead. (This is a *great* simplification, but will serve to illustrate the steps necessary to predict the output of this sensor.)



Envision a column of our sensor's pixels' GSDs moving across the ground in the along-track direction. When the leading edge of the column is over the point source, call that time t_0. As the column moves across the point source at speed $v_x \approx \dfrac{2\pi R_E}{P} \approx 7.386\text{ km}\cdot\text{s}^{-1}$, it moves a distance $x = v_x(t - t_0)$ in time interval $\Delta t = t - t_0$. Setting $t_0 = 0$ gives just $x = v_x t$. Also, the point source is at a distance $x^* = -x + \frac{IFOV}{2} = -v_x t + \frac{IFOV}{2}$ from the nadir point of the column of pixels.

Ignoring the division of the column into 16 separate GSDs for the moment ~ and ignoring the sensor's duty cycle (treating the detectors' output as continuous) ~ the <u>total</u> output *rate* from the column of detectors is

$$\dot{N} \approx \frac{0.838\, I_0}{R^2} \tau_{ATM} A_R \cos\theta\, \tau_{OPT} \frac{\eta}{hc/\bar{\lambda}} \left[\mathrm{s}^{-1}\right].$$



But this expression must be looked at a little more closely: from the figure at right, we see that $R^2 = (x^*)^2 + z^2 = \left(-v_x t + \frac{IFOV}{2}\right)^2 + z^2$. One could certainly argue that the maximum value of *x** is only 41.89 km, giving a maximum range from the sensor to the end of its pixel column of 302.91 km, or only about a 0.97 % increase, but due to the nature of this problem, it is necessary to introduce the element of time through this parameter.

Continuing to look more closely at sensor output, the diagram shows that we even have

$$\frac{1}{R^2}\cos\theta = \frac{1}{R^2}\cdot\frac{z}{R} = \frac{z}{\left[\left(-v_x t + \frac{IFOV}{2}\right)^2 + z^2\right]^{3/2}}.$$

Furthermore, a proper rendering of the atmospheric transmission term is

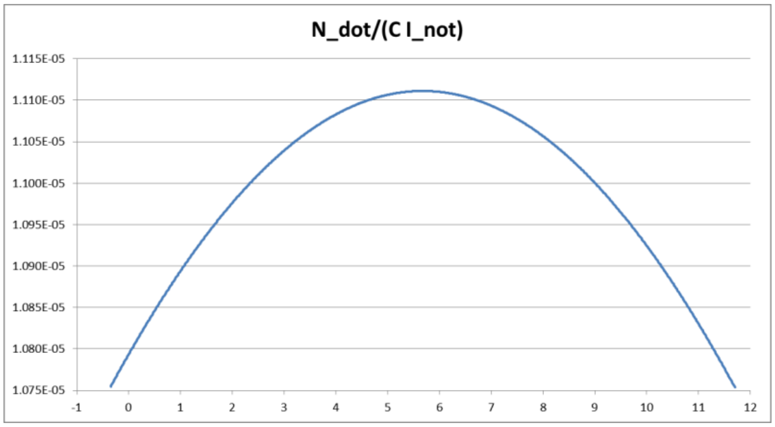
$$\tau_{ATM} = \left[\tau_{ATM}(\mathrm{Nadir})\right]^{1/\cos\theta} = \left[\tau_{ATM}(\mathrm{Nadir})\right]^{\sqrt{\left(-v_x t + \frac{IFOV}{2}\right)^2 + z^2}\Big/ z}.$$

For reasonable values of vertical transmission, $0.5 \le \tau_{ATM}(\mathrm{Nadir}) \le 1.0$, this formidable-looking function varies by no more than 0.68 % at the ends of the pixel column $(\theta = 8°)$, so we can probably get away with neglecting it in the remainder of this calculation.

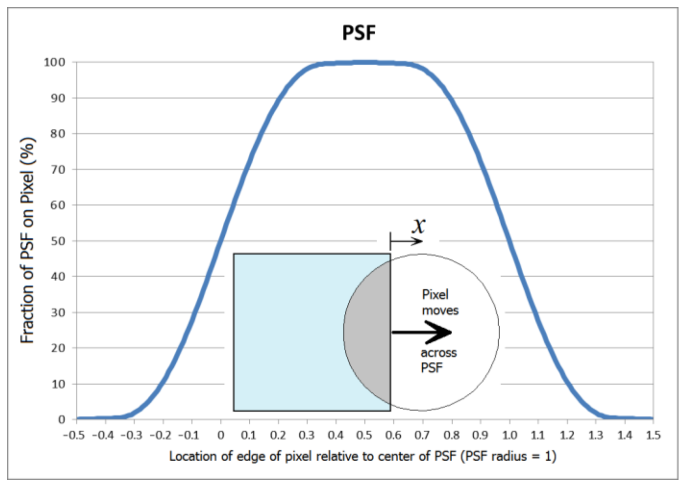
Ignoring the front and back end effects of the detector column passing over the point source, we can now write the sensor's output rate as

$$\dot{N} \approx \frac{0.838 \tau_{ATM} A_R \tau_{OPT} \bar{\lambda} \eta}{hc} \cdot \frac{z}{R^3} I_0 \approx C \frac{z}{\left[\left(-v_x t + \frac{IFOV}{2}\right)^2 + z^2\right]^{3/2}} I_0$$

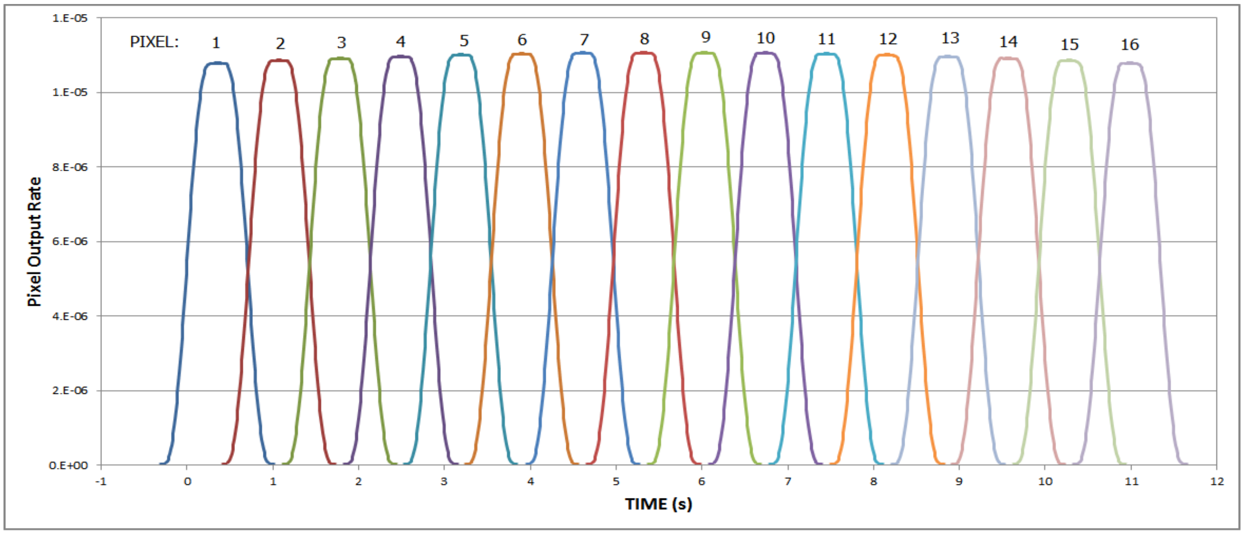
where all the constants have been lumped together into one coefficient, "*C*". Not only is this valid for times $0 \le t \le \frac{IFOV}{v_x} \approx 11.342\ \mathrm{s}$ (when the center of the PSF is on the column), but we must extend it to approximately $-0.354\ \mathrm{s} \le t \le 11.697\ \mathrm{s}$ to cover the interval from when the PSF first touches the column of pixels to when the PSF last touches it. (Remember, time is tracking the center of the PSF, and t =0 is when it is on the leading edge of the pixel row.) Dividing by the constant and the intensity, the value of $\frac{\dot{N}}{CI_0}$ is plotted on the next page as a function of time (see the spreadsheet for the calculation).



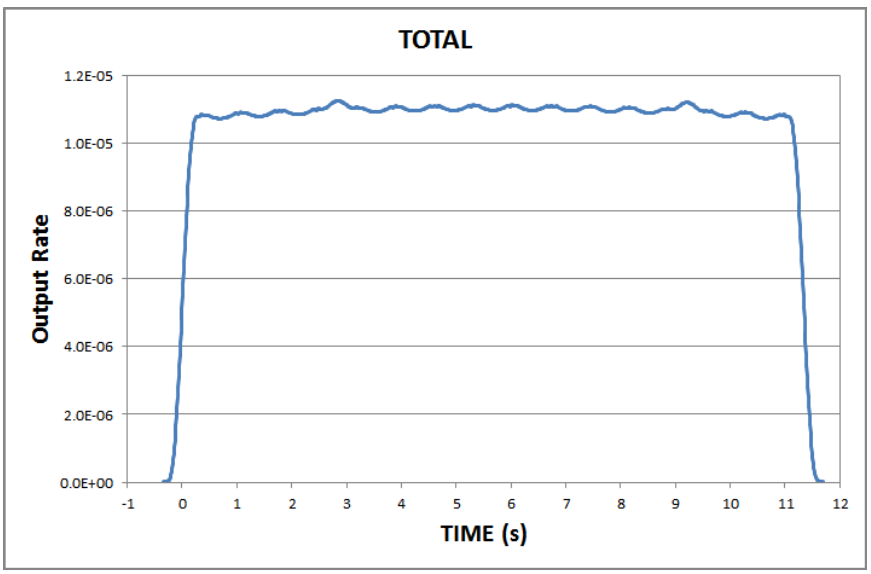
Our next consideration is to think about the PSF passing across the pixels in the column. The output of an individual pixel will be due only to that fraction of the PSF falling on its surface, which can be found from the convolution of the Airy disk with a pixel, shown as the gray shaded area in the figure inset, at right. (The figure is from a computer simulation rather than a calculation, and is not given in the companion spreadsheet; full details can be furnished by contacting the author (HE).) Recall from above that the coordinate, x, is measured to the left from the leading edge of the advancing pixel.

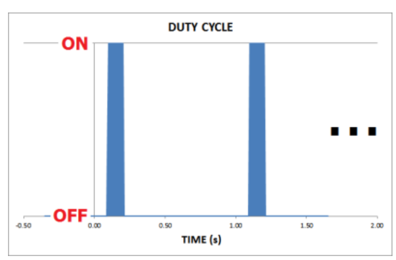


The convolution multiplied by the continuous output rate of the column is the continuous output rate of a pixel. This is shown in the next figure for all sixteen pixels as a function of time, where the output rate is the normalized rate (divided by the constant and I_0) shown above. (The calculation is rather tedious. Again refer to the spreadsheet to see it.)



Next, the total column output rate is the sum of its pixels. This is shown in the next figure, and it should look like the one above (but different scale on the vertical axis, where we're showing the zero on this one). The lumpiness on the top is due to the coarseness of the calculation, but the slight rise in the middle due to the range effect can be clearly seen.





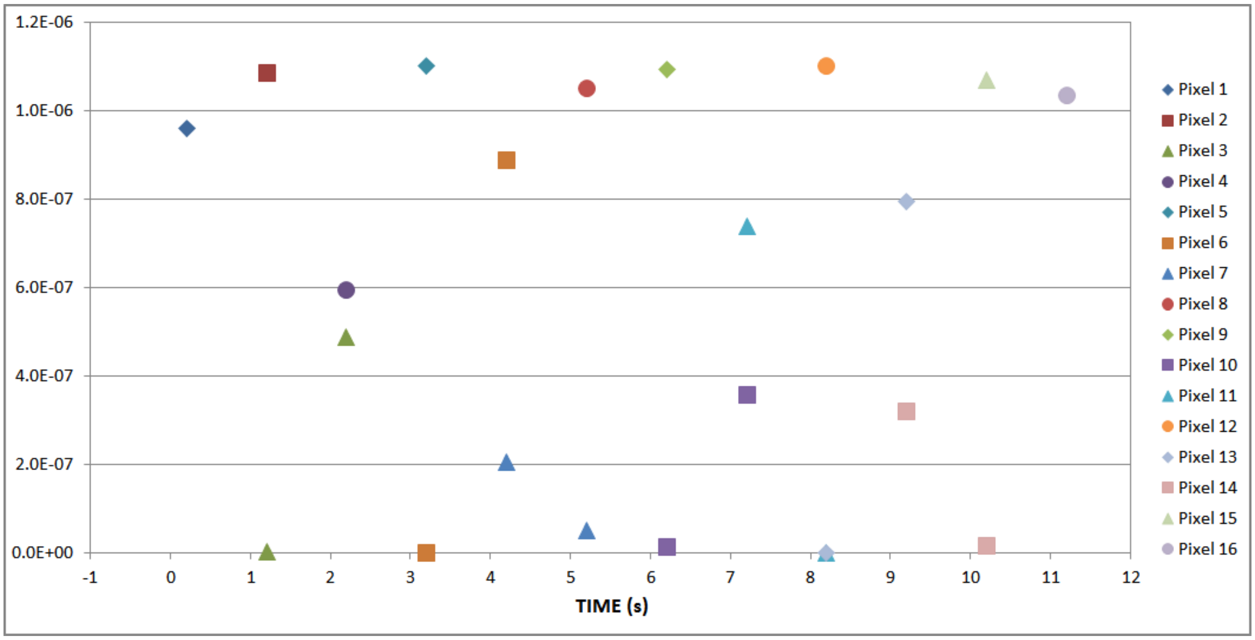
The final step in this part of the problem is to apply the duty cycle. (A 10% duty cycle does not make full use of a pushbroom sensor's collection capabilities, but the next part of this problem will explain the reasoning.) The time when the sensor begins integrating compared to when the target is in view, which we have called the *phasing*, is arbitrary, so a random time was selected. For the following demonstration, the random time is 0.10 s; that is, Δt_{INT} is for $0.10\,\text{s} \le t \le 0.20\,\text{s}$ every second. The sketch illustrates this.

The sensor's temporal integration then gives the output as

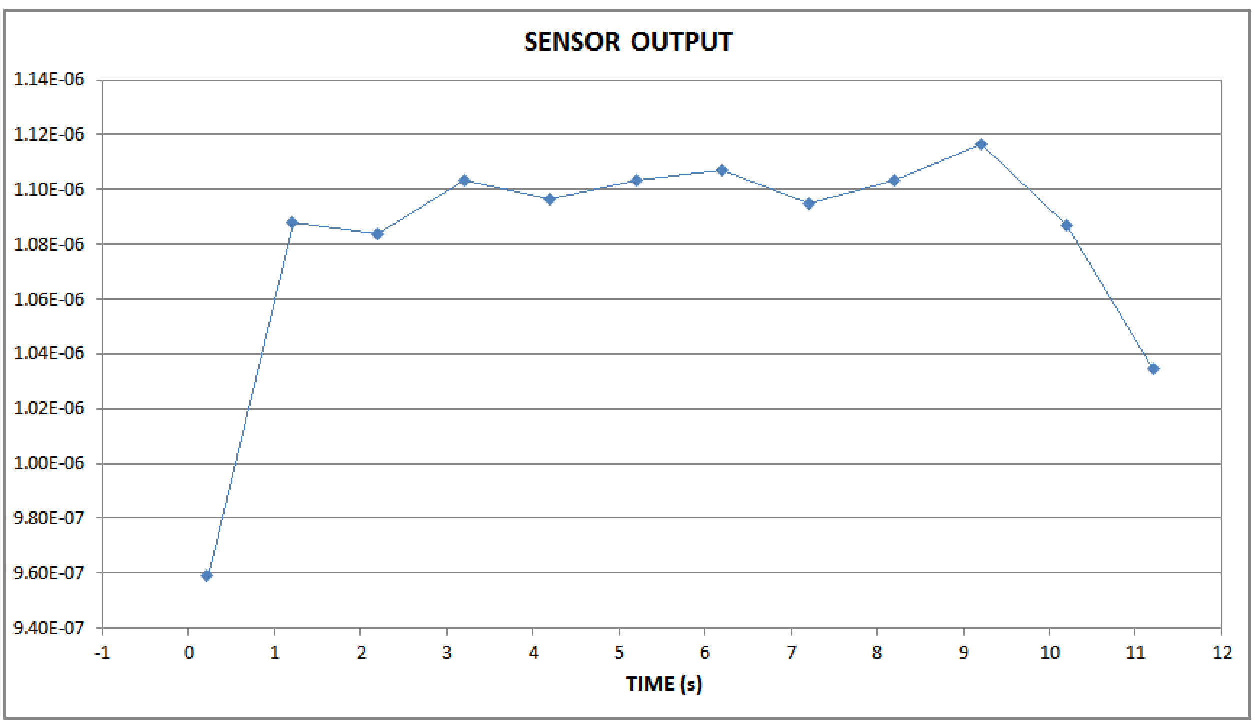
$$N = \int_{\Delta t_{INT}} \dot{N} dt \, ,$$

but we will be calculating the "normalized" sensor output $\frac{N}{CI_0}$. This is again quite tedious.

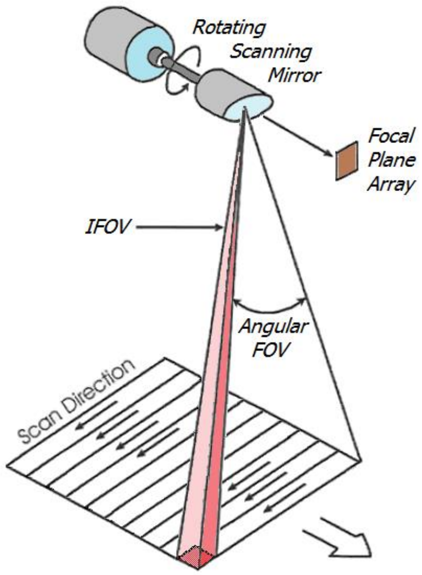
First, we compute the output of each pixel as shown here, noting that the low duty cycle allows each pixel to sample the target only a few times, three at most but only once for some pixels.



Lastly, we add the output of the individual pixels to show the sensor's total output.



This plot resembles the basic calculation, but shows some variance because of the roughness of the calculation. The most noticeable departures are the start and end, where the target is only partially on the column.

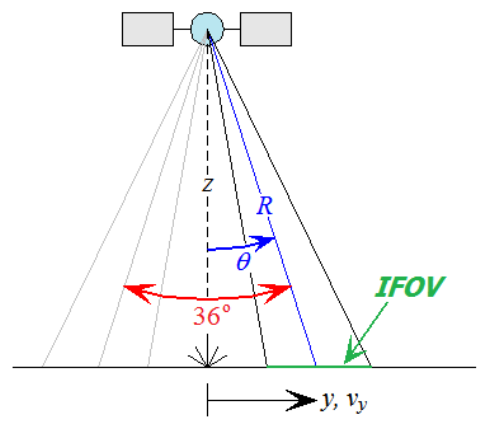


C. If the description of the sensor's collection in this part of the problem is puzzling, see the drawing at left. As the sensor moves in the *x*-direction, the spinning mirror causes its IFOV to rotate, scanning across the ground in the cross-track, or *y*-direction. At a spinning rate of

$$f_{ROT} = 1.0\text{ Hz, or } \omega = 360°\,\text{s}^{-1} = 2\pi\,\text{s}^{-1},$$

and with a 10% duty cycle, the sensor is turned on while it scans through

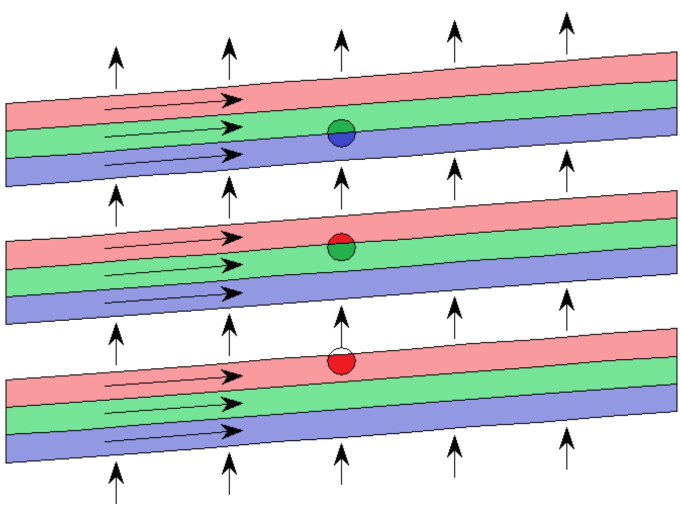
$$\theta = \omega t = \left(360°\,\text{s}^{-1}\right)\left(0.1\,\text{s}\right) = 36°.$$



We can imagine that this is arranged such that the sensor views the ground symmetrically about its nadir. This is illustrated in the next drawing (at right), where the view is from the back; that is, the sensor is moving in the *x*-direction away from you. At nadir, the rate that the IFOV is moving across the ground is

$$v_y = z\omega = \left(300\text{ km}\right)\left(2\pi\,\text{s}^{-1}\right) \approx 1885\text{ km}\cdot\text{s}^{-1},$$

which is approximately 255 times faster than the along-track speed.



Our next drawing shows notionally how the pixels in one column of the FPA sweep across our target. (The target is stationary and the sensor is moving up the page while scanning from left to right. The red, green, and blue bands represent the successive paths of the same pixel.) The situation is not any different than that given above (Part B), except that now each pixel sees the target for a shorter interval of time. With our hypothetical target's PSF being approximately 1° wide (filling only one pixel), it will be within the GSD of one pixel for about

$$\frac{1°}{36°\text{scan}^{-1}} \times 0.10\,\text{s}\cdot\text{scan}^{-1} \approx 0.00278\,\text{s}.$$

Thus the expected output *per pixel* will be only a fraction (1/36$^{\text{th}}$) of that calculated above. However, the FPA is 16 pixels wide. If we specify that the sensor pre-processes its collected data on-board to add the output of all 16 pixels in a row on each scan, then the output per scan will be qualitatively the same as that calculated for Part B, but quantitatively only approximately 16/36 ≈ 44% as much.