SUGGESTED SOLUTIONS (ODD)

CHAPTER 10

NOTE: Use three-digit precision for all calculations unless otherwise stated or implied.

10-1. *A Proposed Forest Fire Detector.* Suppose a US Forest Service sensor is flown on a platform in a circular orbit at an altitude of $H \approx 450\text{ km}$. One channel of the sensor operates in the $8.00 < \lambda < 12.00\ \mu\text{m}$ "thermal" bandpass. The sensor's Newtonian telescope has a collecting mirror of diameter $D \approx 0.300\text{ m}$, reflectivity $\rho \approx 0.950$, and focal length $f \approx 1.00\text{ m}$. The detector is Hg:Cd:Te with a quantum efficiency of $\eta \approx 0.350$, and operates with an integration time of $\Delta t_{\text{int}} \approx 1.00\text{ s}$. Each pixel on its focal plane has a square footprint of $\text{GSD} \times \text{GSD} \approx 1.00 \times 1.00\text{ km}$. (Compare this to the sensor's theoretical spatial resolution.) Assume average atmospheric transmission in-band is $\tau_{ATM} \approx 0.950$ due primarily to absorption.

A. Calculate a pixel's output when it views a uniformly radiant forest on Earth's surface having temperature $T_{TREE} \approx 285\text{ K}$ and reflectivity $\rho_{TREE} \approx 0.100$.

B. Suppose a careless camper has started a small forest fire (less than one pixel) burning with temperature $T_{FIRE} \approx 800\text{ K}$ and emissivity $\epsilon_{FIRE} \approx 0.800$. Further suppose that the fire's radiation (only) is extinguished by smoke particles, $\tau_{SCAT} \approx 0.500$. What fraction of the IFOV of one pixel is ablaze if its output has increased by 1% from its forest-only value?

SUGGESTED SOLUTION: Before getting to the meat of this problem, there are a couple of preliminary calculations to be done. **First**, according to this sensor's specifications, the minimum resolvable angle (Rayleigh criterion) between two point sources (assuming diffraction limited) is $\theta_{MIN} \approx \dfrac{1.22\lambda}{D} = \dfrac{(1.22)(12\times10^{-6}\text{m})}{(0.3\text{ m})} \approx 4.88\times10^{-5}\text{ rad}$, at worst, for the longest wavelength in the bandpass. On the Earth this would be a physical separation of $X = \theta_{MIN} h \approx (4.88\times10^{-5})(4.5\times10^{5}\text{m}) \approx 22\text{ m}$. Since we are told that GSD is 1 km, it is impossible to satisfy the Nyquist criterion for spatial resolution. Therefore we understand that this sensor is strictly "non-imaging" and we can only rely on its output for signature interpretation.

Second, the period[1] of this sensor is $P \approx \sqrt{(9.895\times10^{-5}\text{ s}^2\text{km}^{-3})(6370\text{ km} + 450\text{ km})^3} \approx 5603\text{ s}$, and its IFOV presumably travels $2\pi R_E \approx 40,024\text{ km}$ per orbit. The speed of the field of view over the ground is therefore $40,024\text{ km} \div 5603\text{ s} \approx 7.14\text{ kms}^{-1}$. So in one second, a pixel

[1] This is Kepler's Third Law, $P^2 = Ka^3$, which is covered in Chapter 11. "P" is the orbital period and "a" is the semi-major axis of the orbit (or radius for a circular orbit). For near-Earth orbits, $a \approx R_E + H$ where $R_E \approx 6370$ km. The constant $K \approx 9.895\times10^{-5}\text{ s}^2\text{km}^{-3}$ when P is in seconds and a is in kilometers.

receives photons from a strip of ground approximately[2] 1 km × 8.14 km. That is, the problem does not imply that the sensor has any kind of motion compensation.

A. When a pixel sees nothing but uniform forest in its IFOV (even with motion – photons from the strip of ground arrive at the sensor headed for the same pixel) its output should be ~

$$
\begin{aligned}
N_{PIX}^{TREES} &\approx \int_{8}^{12} L_{\lambda}^{TREES} \Omega_{PIX} \tau_{ATM} A_R \cos\theta_R \, \tau_{OPT} \frac{\eta \mathrm{F}}{hc/\lambda} d\lambda \, \Delta t_{INT} \\
&\approx \int_{8}^{12} \frac{\grave{o}^{TREES} B_{\lambda}^{TREES}}{\pi} \left(\frac{(\mathrm{GSD})^2}{H^2} \right) \tau_{ATM} \left(\pi \left(\frac{D}{2} \right)^2 \right) \cos\theta_R \, \tau_{OPT} \, \eta \mathrm{F} \, \frac{\lambda}{hc} d\lambda \, \Delta t_{INT} \\
&\approx \frac{(1-\rho^{TREES})(\mathrm{GSD})^2 \tau_{ATM} D^2 \cos\theta_R \, \tau_{OPT} \, \eta \mathrm{F} \;\; \Delta t_{INT}}{4H^2 hc} \int_{8}^{12} B_{\lambda}^{TREES} \lambda \, d\lambda \\
&\approx \frac{(1-0.1)(1\,\mathrm{km})^2 (0.95)(0.3\,\mathrm{m})^2 (1)(.95)(.35)(1)(1\,\mathrm{s})}{(4)(450\,\mathrm{km})^2 (6.626\times 10^{-34}\,\mathrm{J\,s})(2.998\times 10^{8}\,\mathrm{m\,s^{-1}})} \left[939.5 \frac{\mathrm{W\,\mu m}}{\mathrm{m}^2} \right] \left(\frac{10^{-6}\,\mathrm{m}}{\mu\mathrm{m}} \right) \\
&\approx 1.49 \times 10^{14} \text{ electrons per frame}
\end{aligned}
$$

where we have done the integral for a 285 K blackbody on the companion spreadsheet. Note however that the integral has an extra factor of wavelength in it, so an extra micron unit pops out. To correct for this, we have to multiply by the appropriate conversion factor at the end.

B. Now let us say that a fraction, *F*, of a pixel's IFOV swath is covered by a forest fire with the parameters stated in the problem. Essentially, this gives the pixel an output equal to the sum of the non-burning fraction of forest in its IFOV plus the fire as follows. (Aside from the area difference and the temperatures and emissivities, the only other consideration is the addition of scattering above the fire from the smoke it produces.)

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[2] Why is this 1 km × 8.14 km and not 1 km × 7.14 km? It is because the back of a pixel's IFOV moves forward the 7.14 km during one integration time, but the front projects out another km ahead.

$$
\begin{aligned}
N_{PIX}^{TOTAL} &= (1-F)N_{PIX}^{TREES} + FN_{PIX}^{FIRE} \\
&= (1-F)N_{PIX}^{TREES} + F\int_{8}^{12} L_{\lambda}^{FIRE}\Omega_{PIX}\tau_{ATM}\tau_{SCAT}A_R\cos\theta_R\,\tau_{OPT}\,\frac{\eta\mathrm{F}}{hc/\lambda}\,d\lambda\,\Delta t_{INT} \\
&= (1-F)N_{PIX}^{TREES} + F\left\{\frac{\partial^{FIRE}\left(\mathrm{GSD}\right)^2\tau_{ATM}\tau_{SCAT}D^2\cos\theta_R\,\tau_{OPT}\,\eta\mathrm{F}\;\Delta t_{INT}}{4H^2hc}\int_{8}^{12}B_{\lambda}^{FIRE}\lambda\,d\lambda\right\} \\
&\approx (1-F)\left(1.49\times10^{14}\right) \\
&\quad + F\left\{\frac{(0.8)\left(1\,\mathrm{km}\right)^2(0.95)(0.5)(0.3\,\mathrm{m})^2(1)(.95)(.35)(1)(1\mathrm{s})}{(4)(450\,\mathrm{km})^2(6.626\times10^{-34}\mathrm{J\,s})(2.998\times10^{8}\mathrm{m\,s^{-1}})}\left[30,490\,\frac{\mathrm{W\,\mu m}}{\mathrm{m}^2}\right]\left(\frac{10^{-6}\mathrm{m}}{\mu\mathrm{m}}\right)\right\} \\
&\approx (1-F)\left\{1.49\times10^{14}\right\} + F\left\{2.15\times10^{15}\right\} \overset{SET}{=} (1.10)(1.49\times10^{14})
\end{aligned}
$$

As indicated, the last step in our problem solution is to set our answer equal to 10% more than what we had in Part A. The fraction of a pixel's swath that needs to be covered with fire turns out to be quite small:

$$
F \approx \frac{(0.01)(1.49\times10^{14})}{2.15\times10^{15} - 1.49\times10^{14}} \approx 0.000745\,.
$$

QUESTION: How big an area is this?

ANSWER: Since a pixel's swath (per integration period) covers an area of 1 km × 8.14 km, this small fraction amounts to only about 6.06×10^{-3} km^2, or an area about 78 × 78 m. (This is approximately the size of a regulation soccer field, 100 m × 60 m.) Our answer is remarkable, but certainly what the US Forest Service would want from such an overhead forest fire detecting sensor. Another problem, of course, is that such a sensor would not have a frequent enough revisit rate to any given forest to provide anything close to continuous warning.

10-3. *Counting Polar Bears*. In response to an RFP, we propose the arctic polar bear population be counted by a Polar Orbiting Osological Habitat (POOH) sensor. POOH is to orbit at 100 km, have a 1°×1° FOV, and sense 11± 0.5 μm photons. From altitude, the cold arctic landscape (T ≈ –40°C) appears roughly uniform with an albedo of 50%.

Polar bears, on the other paw, are warm-blooded critters (T ≈ 40°C), and their fur is approximately 5% transmitting in the thermal infrared. (Their fur is closer to ambient than body temperature.) Estimate whether polar bear detection is possible with POOH. (Assume a spherical polar bear with enough fur to make a rug covering ≈ 10 m^2 of floor space.)

SUGGESTED SOLUTION: First, we'll calculate the expected output of the POOH sensor when it sees the Arctic background. Our assumption is – working in the thermal infrared – there will be little or no reflected sunlight, so this method for detecting polar bears, if it works, should be able to find them in the dark (which is a good thing for above the Arctic Circle in the winter). The appropriate phenomenological end-to-end equations for the background is

$$\dot{N}_{ARCTIC} = \int_{BANDPASS} L_\lambda \Omega \cos\theta_R \tau_{ATM} A_R \tau_{OPT} \frac{\mathrm{F}\,\eta}{hc/\lambda} d\lambda$$

To simplify this calculation, we will assume that $\cos\theta_R \approx 1$, and that τ_{ATM}, τ_{OPT}, and η are all wavelength independent. Then since there are several sensor parameters in this equation that are unspecified, but are the same for both background and bears, we will divide them out to make a reduced equation ~

$$\dot{N}^*_{ARCTIC} = \frac{\dot{N}_{ARCTIC}}{\tau_{ATM} A_R \tau_{OPT} \mathrm{F}\,\eta/hc} = \int_{BANDPASS} L_\lambda \Omega \lambda\, d\lambda$$

$$= \int_{BANDPASS} \frac{(1-\rho_{ALBEDO}) B_\lambda(233\,\mathrm{K})}{\pi} \frac{\pi\alpha^2}{4} \lambda\, d\lambda = \frac{(1-\rho_{ALBEDO})\alpha^2}{4} \int_{10.5\,\mu m}^{11.5\,\mu m} \lambda B_\lambda d\lambda$$

$$= \frac{(1-0.50)\left[1^\circ \times \frac{\pi}{180^\circ}\right]^2}{4} (93.7\ \text{W-}\mu\text{m/m}^2) \approx 3.57\times10^{-3}\ \text{W-}\mu\text{m/m}^2$$

where we have used the companion spreadsheet to numerically compute the integral (this should be routine by now – we've used the same type of calculation for the last three problems).

Second, and in a similar vein, we calculate the expected output of our POOH sensor when it sees a polar bear, assuming a bear is a point source ~

$$\dot{N}_{BEAR} = \int_{BANDPASS} \frac{I_\lambda}{R^2} \cos\theta_R \tau_{ATM} A_R \tau_{OPT} \frac{\mathrm{F}\,\eta}{hc/\lambda} d\lambda$$

We will treat this formula in the same manner as the last one, and according to the problem statement, this end-to-end equation will be for 5% of the bear's body heat transmitting through its fur ~

$$\dot{N}^*_{BEAR} = \frac{\dot{N}_{BEAR}}{\tau_{ATM} A_R \tau_{OPT} \mathrm{F}\,\eta/hc} = \int_{BANDBASS} \frac{I_\lambda}{R^2} \lambda\, d\lambda$$

$$= \int_{BANDPASS} \frac{t_{FUR} B_\lambda(313\,\mathrm{K}) A_{BEAR}}{4\pi R^2} \lambda\, d\lambda = \frac{t_{FUR} A_{BEAR}}{4\pi R^2} \int_{10.5\,\mu m}^{11.5\,\mu m} \lambda B_\lambda d\lambda$$

$$= \frac{(0.05)(10\ \mathrm{m}^2)}{(4\pi)(10^5\mathrm{m})^2} (398\ \text{W-}\mu\text{m/m}^2) \approx 1.58\times10^{-9}\ \text{W-}\mu\text{m/m}^2$$

If this is a non-imaging sensor, then with six orders of magnitude difference between background and bears, it looks like our proposal will never fly.

NOTE ADDED: The justification for the 5% transmission of polar bear fur comes from a study by J.A. Preciado, B. Rubinsky, D. Otten, B. Nelson, M.C. Martin, & R. Greif, "Radiative Properties of Polar Bear Hair" (ASME 2002 Advances in Bioengineering, BED Vol. 53), available on-line at http://infrared.als.lbl.gov/pubs/PolarBearASME.pdf. Their Figure 1, is reproduced here, showing that at thermal wavelengths, maybe even 5% is an overestimate. From the paper, the general behavior of polar bear fur seems to be to absorb high energy (ultraviolet) radiation and transmit the energy to the beast to keep him warm. The individual hairs are mostly transparent, but appear white because of visible light scattering in the fur. Polar bears' skin is actually dark as well.

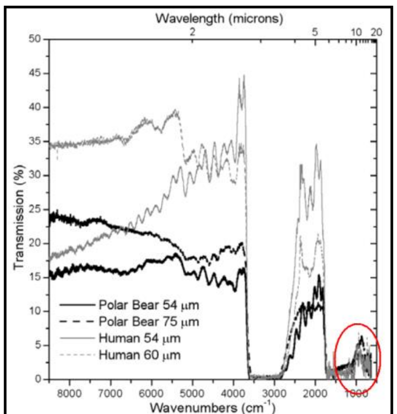

Figure 1. Percent Transmissivity Values versus Wavenumbers for Two Polar Bear Hairs and Human Hairs of Similar Thickness.

10-5. ***An Alien Remote Sensor.*** A (very) remote sensor on a satellite in a 350 km circular orbit around an alien planet has a collecting area of 100 cm^2 and a circular FOV of 0.05 sr. The optical transmission function, the sensor's quantum efficiency, and the transmission through the planet's atmosphere are, respectively,

$$\tau_{OPT} = \begin{cases} 0.9 & \text{for } 0.22\ \mu\text{m} < \lambda < 1.05 \mu\text{m} \\ 0 & \text{for all other } \lambda \end{cases}, \quad \eta = \frac{\lambda}{0.3} e^{-\lambda/0.3} \text{ electrons/photon } (\lambda \text{ in } \mu\text{m}), \text{ and}$$

$$\tau_{ATM} = \begin{cases} 0.8 & \text{for } 0.28\ \mu\text{m} < \lambda < 1.27 \mu\text{m} \\ 0 & \text{for all other } \lambda. \end{cases}$$

The sensor sees a uniform extended surface, filling its FOV, having a spectral radiance of

$$L_\lambda = \begin{cases} 0 & & \text{for } \lambda < 0.2\ \mu\text{m} \\ \dfrac{2\times 10^{-7}}{\lambda} & \text{W}\cdot\text{cm}^{-2}\text{sr}^{-1}\mu\text{m}^{-1} & \text{for all other } \lambda \ (\lambda \text{ in } \mu\text{m}) \end{cases}$$

A. What is the spectral bandpass of this sensor?

B. What is the irradiance on its aperture from the source (in the sensor's bandpass)? Express this answer in both watts per square centimeter and photons per second per square centimeter.

C. What is the power on the focal plane of the sensor (in the sensor's bandpass)? Express this answer in both watts and photons/second.

D. What is the electron count rate from the focal plane due to this source?

SUGGESTED SOLUTION: **A.** The bandpass of this sensor is defined by the optical transmission function: $0.22 \le \lambda \le 1.05$ μm. This is shown in the companion spreadsheet where the two wavelength-dependent sensor parameters, optical transmission and quantum efficiency, are calculated and plotted. (Note that the atmosphere does not transmit light at wavelengths shorter than 0.28 μm. This will have an influence on our calculations, but does not impact the bandpass of the sensor itself.)

B. There are two ways to do this part of the problem – numerically and analytically. First, we can tackle it numerically like we have done several times before. The irradiance on the sensor's aperture, in "engineering units" is

$$\mathrm{E} = \int_{BANDPASS} \mathrm{E}_\lambda d\lambda = \int_{0.22\,\mu\mathrm{m}}^{1.05\,\mu\mathrm{m}} L_\lambda \Omega \cos\theta_R \tau_{ATM}\, d\lambda \approx \Omega \tau_{\mathrm{ATM}} \int_{0.28\,\mu\mathrm{m}}^{1.05\,\mu\mathrm{m}} L_\lambda d\lambda$$
$$\approx (0.05\,\mathrm{sr})(0.8)(2.66\times10^{-7}\,\mathrm{W\text{-}cm^{-2}sr^{-1}}) \approx 1.06\times10^{-8}\,\mathrm{W/cm^2}$$

where the integral has been evaluated in the companion spreadsheet (Column H), and we have ignored the cosine of the fixation angle. Our justification for the latter comes from calculating ~

$$\Omega = 0.05\, sr \approx \frac{\pi\alpha^2}{4} \quad \Rightarrow \theta_R^{(MAX)} = \frac{\alpha}{2} \approx 7.23^\circ \quad \Rightarrow \cos\theta_R^{(MAX)} \approx 0.992$$

Also note, by the way, that there is no contribution to the irradiance between $0.22 \le \lambda < 0.28$ μm due to the atmosphere. This changes the limits on the integral, and the atmospheric term itself comes out of the integral because it is a constant.

The photon irradiance on the sensor's aperture is found similarly by dividing the surface spectral radiance by the energy per photon ~

$$\mathrm{E}_{PHOTON} = \int_{0.22\,\mu\mathrm{m}}^{1.05\,\mu\mathrm{m}} \frac{L_\lambda}{hc/\lambda} \Omega \cos\theta_R \tau_{ATM}\, d\lambda \approx \Omega \tau_{ATM} \int_{0.28\,\mu\mathrm{m}}^{1.05\,\mu\mathrm{m}} \frac{L_\lambda}{hc/\lambda} d\lambda$$
$$= (0.05\,\mathrm{sr})(0.8)\left(7.75\times10^{11}\,\frac{\mathrm{photons}}{\mathrm{s\cdot cm^2 sr}}\right) = 3.10\times10^{10}\,\frac{\mathrm{photons}}{\mathrm{s\cdot cm^2}}$$

Both of these answers can be gotten analytically by actually doing the integrals (for those who know how). For the irradiance,

$$\mathrm{E} = \int_{0.22\,\mu\mathrm{m}}^{1.05\,\mu\mathrm{m}} L_\lambda \Omega \cos\theta_R \tau_{ATM}\, d\lambda \approx \Omega\tau_{ATM} \int_{0.28\,\mu\mathrm{m}}^{1.05\,\mu\mathrm{m}} L_\lambda\, d\lambda = \Omega\tau_{ATM} \int_{0.28\,\mu\mathrm{m}}^{1.05\,\mu\mathrm{m}} \frac{2\times10^{-7}}{\lambda}\, d\lambda$$

$$= \Omega\tau_{ATM}(2\times10^{-7}\,\mathrm{W\text{-}cm^{-2}sr^{-1}})\left[\ln\lambda\right]_{0.28\,\mu\mathrm{m}}^{1.05\,\mu\mathrm{m}} = (0.05\,\mathrm{sr})(0.8)(2\times10^{-7}\,\mathrm{W\text{-}cm^{-2}sr^{-1}})(1.322)$$

$$\approx 1.06\times10^{-8}\,\mathrm{W/cm^2}.$$

Not surprisingly, we got the same answer. The only caution with this solution is watching out for the units. The constant in the spectral radiance (call it $c^* = 2\times10^{-7}$) carries the units of watts per square centimeter per steradian, and the natural logarithm is unitless.[3]

And for the photon irradiance, the answer is a particularly simple integral, but a morass of units to wade through; although the wavelength dependence appears to cancel out, there is a residual conversion as shown here:

$$\mathrm{E}_{PHOTON} = \int_{0.22\mu\mathrm{m}}^{1.05\mu\mathrm{m}} \frac{L_\lambda \Omega\cos\theta_R \tau_{ATM}}{hc/\lambda}\, d\lambda \approx \Omega\tau_{ATM} \int_{0.28\mu\mathrm{m}}^{1.05\mu\mathrm{m}} \frac{c^*/\lambda}{hc/\lambda}\, d\lambda = \frac{\Omega\tau_{ATM} c^*}{hc} \int_{0.28\mu\mathrm{m}}^{1.05\mu\mathrm{m}} \left[\frac{\lambda(\mathrm{m})}{\lambda(\mu\mathrm{m})}\right] d\lambda$$

$$= \frac{\Omega\tau_{ATM} c^*}{hc}\left[\frac{\lambda(\mathrm{m})}{\lambda(\mu\mathrm{m})}\right]\Delta\lambda = \frac{(0.05\,\mathrm{sr})(0.8)\left(2\times10^{-7}\,\dfrac{\mathrm{W}}{\mathrm{cm^2 sr}}\right)\left[\dfrac{1\mathrm{m}}{10^6\mu\mathrm{m}}\right](0.77\,\mu\mathrm{m})}{(6.63\times10^{-34}\,\mathrm{J\cdot s})\left(3.00\times10^8\,\dfrac{\mathrm{m}}{\mathrm{s}}\right)}$$

$$= 3.10\times10^{10}\,\frac{\mathrm{photons}}{\mathrm{s\cdot cm^2}}.$$

C. Because the optical transmission function is a constant not dependent on wavelength, finding the in-band power and number of photons per second passed to the focal plane array is

$$\Phi_{FPA} = \mathrm{E}\, A_R \tau_{OPT} = (1.06\times10^{-8}\,\mathrm{W/cm^2})(100\,\mathrm{cm^2})(0.9) \approx 9.54\times10^{-7}\,\mathrm{W}$$

and

$$\dot{N}_{PHOTON} = \mathrm{E}_{PHOTON} A_R \tau_{OPT} = \left(3.10\times10^{10}\,\frac{\mathrm{photons}}{\mathrm{s\cdot cm^2}}\right)(100\,\mathrm{cm^2})(0.9) \approx 2.79\times10^{12}\,\frac{\mathrm{photons}}{\mathrm{s}}$$

D. As you can tell by now, we are building up the solution to this problem piece-by-piece. Now that we've got the power (and photons) at the focal plane, we want our photo-detector array to turn it into electron output rate. That's easy ~ the end-to-end equation should be familiar ~

$$\dot{N} \approx \int_{BANDPASS} L_\lambda \Omega \cos\theta_R\, \tau_{ATM}\, A_R\, \tau_{OPT}\, \frac{\eta}{hc/\lambda}\, d\lambda = \frac{\Omega\cos\theta_R\, \tau_{ATM}\, A_R\, \tau_{OPT}}{hc} \int_{BANDPASS} L_\lambda \lambda \eta\, d\lambda$$

~ but introduction of the quantum efficiency again requires us to be a little careful. For the quantum efficiency to be dimensionless (well, "electrons per photon" technically, but that's not a

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[3] In detail: $\left[\ln\lambda\right]_{0.28\mu m}^{1.05\mu m} = \ln(1.05\,\mu m) - \ln(0.28\,\mu\mathrm{m}) = \ln\left(\dfrac{1.05\,\mu m}{0.28\,\mu\mathrm{m}}\right) = \ln(3.75) = 1.322$

unit) the coefficients "0.3" in the denominator of the lead term, and in the denominator of the exponent, must both have units of microns if we're going to express wavelength in microns. Doing the calculation in microns makes the most sense, so tracking through the units shows us that we're going to need to include a microns-to-meters equivalence factor ~

$$\dot{N} \approx \frac{\left(2\times10^{-7}\,\dfrac{\mathrm{W}}{\mathrm{cm^2 sr}}\right)(0.05\,\mathrm{sr})(0.8)\left(100\,\mathrm{cm}^2\right)(0.9)}{\left(6.626\times10^{-34}\,\mathrm{J\cdot s}\right)\left(2.998\times10^{8}\,\dfrac{\mathrm{m}}{\mathrm{s}}\right)(0.3\,\mu\mathrm{m})}\left[\frac{1\mathrm{m}}{10^6\mu\mathrm{m}}\right]\int_{BANDPASS}\frac{1}{\lambda[\mu\mathrm{m}]}\lambda[\mu\mathrm{m}]\lambda[\mu\mathrm{m}]e^{-\lambda[\mu\mathrm{m}]/0.3\mu\mathrm{m}}d\lambda[\mu\mathrm{m}]$$

$$\approx\left(1.208\times10^{13}\mu\mathrm{m}^{-2}\mathrm{s}^{-1}\right)\int_{0.28\mu\mathrm{m}}^{1.05\mu\mathrm{m}}\frac{1}{\cancel{\lambda}}\cancel{\lambda}\,\lambda e^{-\lambda/0.3}d\lambda$$

where we only need to evaluate the integral, remembering that the limits are 0.28 – 1.05 μm. (Note that the numerical factor as calculated in the last equation properly has units of $\mu\mathrm{m}^{-2}\mathrm{s}^{-1}$ while the integral has the units of $\mu\mathrm{m}^2$.) The integral can be done numerically (≈ 0.562) as in the companion spreadsheet, or the brave-hearted of you can do it "by parts" ~

$$\int\lambda e^{-\lambda/0.3}d\lambda=-0.3\lambda e^{-\lambda/0.3}+\int0.3e^{-\lambda/0.3}d\lambda$$

$$=-0.3\lambda e^{-\lambda/0.3}-0.09e^{-\lambda/0.3}=-0.3(\lambda+0.3)e^{-\lambda/0.3}$$

In either case, the final answer to our problem is ~

$$\dot{N}\approx\left(1.21\times10^{13}\mu\mathrm{m}^{-2}\mathrm{s}^{-1}\right)\left(0.0562\,\mu\mathrm{m}^2\right)\approx6.79\times10^{11}\quad\text{electrons/second}.$$

This answer seems reasonable in light of the fact that we found in part C the flux of photons to the detector is 2.79×10^{12} per second while the average[4] quantum efficiency is around 0.243.

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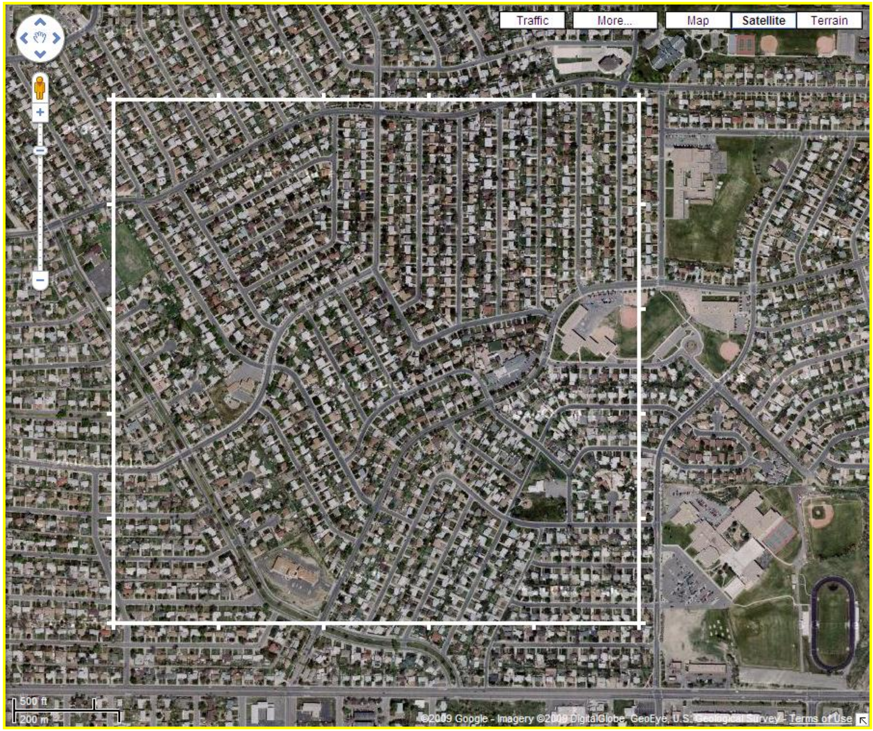
[4] $\langle\eta(\lambda)\rangle=\int\eta(\lambda)d\lambda\Big/\int d\lambda$

10-7. ***Estimating Anthropogenic Light Pollution*.** Some "Earth at Night" pictures are mosaic images collected by weather satellites in their near-IR bands. Estimate the output and noise of such a sensor from (a) one square kilometer of typical Midwestern US suburbia on a dark night (as shown in the Google image below), and (b) compare to the output of one square kilometer of alto-cumulus clouds on a moonlit night. Some sensor parameters are as follows:

Platform:	Orbit: 830 km near-circular, sun-synchronous
Optics:	Aperture: 30 cm diameter, 2.8% obstructed Focal length: 5.25 m (two mirrors, two lenses) Transmission: 81% throughput (all surfaces AR treated) Filter: 78% transmitting in 0.81 – 1.05 μm band
Detector:	Silicon FPA: 1024 × 4096 pixels, 1.22 × 4.92 cm Quantum efficiency: 0.48 in 0.4 – 1.12 μm band Filling factor: 88% Framing & integration: 8 Hz @ 90% DC

Some other hints for putting together your estimates for this problem might be that (1) US tract homes typically have 200 A electrical services (to run all of the appliances we consider to be necessities), (2) the International Dark-Sky Association, www.darksky.org, is trying to minimize the number of lights pointed upwards (but the mean reflectivity of the Earth is ≈ 0.35), and (3) alto-cumulus cloud tops are typically at about 2,000 – 10,000 ft AGL. You may also make use of the fact that the sun is visual magnitude[5] –26.5, while the full moon is only –12.5.

[5] What is a "visual magnitude?" That's the number astronomers use to describe the brightness of an object – smaller numbers (more negative) are brighter, and one visual magnitude is equivalent to $\sqrt[5]{100} \approx 2.51$ times as bright. How does this help? Well, moonlight [not moonshine] is basically reflected sunlight.



SUGGESTED SOLUTION: **(a)** The standard end-to-end equation we want to use for looking at an extended source is ~ $N \approx \int L_\lambda \Omega \tau_{ATM} A_R \tau_{OPT} \tau_{FILTER} \frac{\eta \mathrm{F} \lambda}{hc} \Delta t_{INT} d\lambda$. Considering the hand-waving that is going to follow, there's no need to be precise about the bandpass integral, so we'll just estimate this as ~ $N \approx \frac{1}{hc} \bar{L} \Omega \overline{\tau_{ATM}} A_R \overline{\tau_{OPT}}\, \overline{\tau_{FILTER}}\, \bar{\eta} \bar{\mathrm{F}} \bar{\lambda} \Delta t_{INT}$. One at a time, we'll now work on the terms that go into this equation.

For the surface radiance, start with the Google image and count approximately 1000 houses in the one square kilometer box. Each house typically has a maximum electrical power usage of (200 A)×(110 V) ≈ 2.2×10^4 W, but ordinarily runs at only about 50% of that when the occupants are present (during Prime Time). Let's suppose that about 15% the usage, or 1650 W is being used for lighting. This may seem a little high, but is deliberately so to include any public outdoor lighting (street lamps, parking lot lighting, commercial signage, etc.) that may be adjunct to each house (on the average). Now let's assume that only 20% of the power paid for lighting is for outdoor use ~ about 330 W per household. With any luck at all, this will all be directed downward so that only about 35% of it, 116 W, is reflected skyward. Next, out of the popular variety of lamps in use ~ incandescent, fluorescent, mercury and sodium vapor ~ let's imagine that only 10% of their radiant output is in our sensor's bandpass, $0.81 \le \lambda \le 1.05$ μm (limited by the filter). This gives us 11.6 W of reflected in-band radiant power per household. Then to complete the radiance calculation ~

$$L \approx \frac{1}{\pi} M_{REFLECTED} \approx \frac{1}{\pi} \times \left(\frac{1000\,\text{households}}{10^6\,\text{m}^2} \right) \times \left(\frac{11.6\,\text{W reflected}}{\text{household}} \right) \approx 3.7\times10^{-3} \frac{\text{W}}{\text{m}^2\text{sr}}$$

Since the question asked is to estimate our sensor's output from one square kilometer of suburban sprawl, we'll take $\Omega \approx \dfrac{1\,\text{km}^2}{(830\,\text{km})^2} \approx 1.45\times10^{-6}\,\text{sr}$. The light received through this FOV will fall on multiple pixels, of course, but we needn't worry about how many since we're just going for the total output.

A couple more preliminary calculations and we're ready: first, the aperture receiving photons is $A_R \approx (0.972)\dfrac{\pi(0.3\,\text{m})^2}{4} \approx 6.87\times10^{-2}\,\text{m}^2$ and, second, the integration time is $\Delta t_{INT} = (0.90)\dfrac{1}{8\,\text{Hz}} \approx 0.113\,\text{s}$. We also need to look up the average in-band atmospheric transmission value from a MODTRAN run we've done earlier (we find approximately 0.98), and calculate the average wavelength to be about (0.81 + 1.05 μm)/2 ≈ 0.93 μm. Now we're ready to compute ~

$$
\begin{aligned}
N &\approx \frac{1}{hc}\overline{L}\,\Omega\,\overline{\tau_{ATM}}\,A_R\,\overline{\tau_{OPT}}\,\overline{\tau_{FILTER}}\,\overline{\eta}\,\overline{\mathrm{F}}\,\overline{\lambda}\,\Delta t_{INT} \\
&\approx \frac{\left(3.7\times10^{-3}\dfrac{\text{W}}{\text{m}^2\text{sr}}\right)\left(1.45\times10^{-6}\text{sr}\right)(0.98)\left(6.87\times10^{-2}\text{m}^2\right)(0.81)(0.78)(0.48)(0.88)\left(0.93\times10^{-6}\text{m}\right)(0.113\text{s})}{\left(6.63\times10^{-34}\text{J}\cdot\text{s}\right)\left(3\times10^{8}\dfrac{\text{m}}{\text{s}}\right)} \\
&\approx 5.1\times10^{7}\,\text{electrons.}
\end{aligned}
$$

Finally, for this part, we'll just take noise in the output to be directly proportional to input photon noise, and estimate ~

$$\mathcal{N} \approx \sqrt{2\times5.1\times10^{7}} \approx 1.0\times10^{4}\,\text{electrons per sample}$$

(b) For the alto-cumulus clouds, the only thing we need different from the previous estimate is a value for the in-band radiance. From **Figure 4-21** of the text, we can read off that the spectral radiance of reflected sunlight from objects with reflectivity ρ ≈ 0.9 (as should be the case for nice puffy clouds) is about 300 $\text{W}\cdot\text{m}^{-2}\text{sr}^{-1}\mu\text{m}^{-1}$. The in-band radiance of reflected sunlight should therefore be (300 $\text{W}\cdot\text{m}^{-2}\text{sr}^{-1}\mu\text{m}^{-1}$)×(0.24 μm) ≈ 72 $\text{W}\cdot\text{m}^{-2}\text{sr}^{-1}$. Since the moon is 14 visual magnitudes less bright (and we will assume this is good for the near IR as well), its reflected radiance from the cloud tops should be (72 $\text{W}\cdot\text{m}^{-2}\text{sr}^{-1}$)÷($100^{14/5}$) ≈ 1.8×10^{-6} $\text{W}\cdot\text{m}^{-2}\text{sr}^{-1}$. This is ignoring any attenuation of moonlight through the atmosphere, but we're fairly safe because $\tau_{ATM} \approx 0.98$ is well within our fudge margin. Thus, our sensor's output when looking at moonlit clouds is ~

$$
\begin{aligned}
N &\approx \frac{\left(1.8\times10^{-6}\dfrac{\text{W}}{\text{m}^2\text{sr}}\right)\left(1.45\times10^{-6}\text{sr}\right)(0.98)\left(6.87\times10^{-2}\text{m}^2\right)(0.81)(0.78)(0.48)(0.88)\left(0.93\times10^{-6}\text{m}\right)(0.113\text{s})}{\left(6.63\times10^{-34}\text{J}\cdot\text{s}\right)\left(3\times10^{8}\dfrac{\text{m}}{\text{s}}\right)} \\
&\approx 2.5\times10^{4}\,\text{electrons.}
\end{aligned}
$$

And this time the noise is ~

$$\mathcal{N} \approx \sqrt{2 \times 2.5 \times 10^4} \approx 220 \text{ electrons per sample}.$$

10-9. *Sensitivity to sensor and collection parameters.* Consider an element of a remote sensor's output from a point source:

$$\Delta N \approx \frac{I_\lambda}{R^2} \tau_{ATM} \cos\theta_R \tau_{OPT} \frac{\eta \mathrm{F}}{hc/\lambda} \Delta t_{INT} \Delta\lambda$$

where A_R, τ_{OPT}, η, F, λ, Δt_{INT}, and $\Delta\lambda$ are measured/known sensor hardware/electronic or operational parameters;

R, τ_{ATM}, and θ_R are collection parameters; and

ΔN is the known sensor output.

Spectral intensity, I_λ, is, of course, the unknown. Uncertainty in which of the measured, collection, and known parameters has the greatest influence on the uncertainty in our estimate for solving for I_λ?

SUGGESTED SOLUTION: First, solve the end-to-end equation for spectral intensity as in the text ~

$$I_\lambda \approx \frac{hc\,\Delta N\, R^2}{\tau_{ATM} \cos\theta_R A_R \tau_{OPT} \eta \mathrm{F}\, \lambda \Delta t_{INT} \Delta\lambda}$$

Next, find the variation[6] in spectral intensity implicitly[7] ~

$$\begin{aligned}\delta(I_\lambda) \approx {} & I_\lambda \frac{\delta(\Delta N)}{\Delta N} + 2 I_\lambda \frac{\delta(R)}{R} - I_\lambda \frac{\delta(\tau_{ATM})}{\tau_{ATM}} + I_\lambda \sin\theta_R \frac{\delta(\cos\theta_R)}{\cos\theta_R} - I_\lambda \frac{\delta(A_R)}{A_R} \\ & - I_\lambda \frac{\delta(\tau_{OPT})}{\tau_{OPT}} - I_\lambda \frac{\delta(\eta)}{\eta} - I_\lambda \frac{\delta(\mathrm{F})}{\mathrm{F}} - I_\lambda \frac{\delta(\lambda)}{\lambda} - I_\lambda \frac{\delta(\Delta t_{INT})}{\Delta t_{INT}} - I_\lambda \frac{\delta(\Delta\lambda)}{\Delta\lambda}.\end{aligned}$$

Dividing through by I_λ we have what is known as the fractional variation ~

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[6] In the calculus, the notation "*d*" means an infinitesimal change, while the notation "Δ" means a larger, finite change. For variation, the symbol "δ" is often used, as in this problem, to represent something in between. In particular, this notation usually represents a small uncertainty. The calculus rules for using δ are usually taken to be the same as doing a derivative, or taking a differential.

[7] This variation equation is the result of operating on the spectral intensity like a partial derivative. A typical term is found like this, taking the variation with respect to ΔN ~

$$\begin{aligned}\delta\left(\frac{hc\,\Delta N\, R^2}{\tau_{ATM} \cos\theta_R A_R \tau_{OPT} \eta F\, \lambda \Delta t_{INT} \Delta\lambda}\right) & \approx \left(\frac{hc\, R^2}{\tau_{ATM} \cos\theta_R A_R \tau_{OPT} \eta F\, \lambda \Delta t_{INT} \Delta\lambda}\right) \delta(\Delta N) \\ & \approx \left(\frac{hc\, R^2}{\tau_{ATM} \cos\theta_R A_R \tau_{OPT} \eta F\, \lambda \Delta t_{INT} \Delta\lambda}\right) \left\{\frac{\Delta N}{\Delta N}\right\} \delta(\Delta N) \approx \left(\frac{hc\,\Delta N\, R^2}{\tau_{ATM} \cos\theta_R A_R \tau_{OPT} \eta F\, \lambda \Delta t_{INT} \Delta\lambda}\right)\left(\frac{\delta(\Delta N)}{\Delta N}\right) \approx \mathrm{I}_\lambda \left(\frac{\delta(\Delta N)}{\Delta N}\right).\end{aligned}$$

$$\frac{\delta(I_\lambda)}{I_\lambda} \approx \frac{\delta(\Delta N)}{\Delta N} + 2\frac{\delta(R)}{R} - \frac{\delta(\tau_{ATM})}{\tau_{ATM}} + \sin\theta_R \frac{\delta(\cos\theta_R)}{\cos\theta_R} - \frac{\delta(A_R)}{A_R}$$
$$- \frac{\delta(\tau_{OPT})}{\tau_{OPT}} - \frac{\delta(\eta)}{\eta} - \frac{\delta(\mathrm{F})}{\mathrm{F}} - \frac{\delta(\lambda)}{\lambda} - \frac{\delta(\Delta t_{INT})}{\Delta t_{INT}} - \frac{\delta(\Delta\lambda)}{\Delta\lambda}.$$

Thus we see that the fractional variation (or uncertainty) in our estimate of spectral intensity is directly proportional to the sum of the fractional variations (uncertainties) in all of the variables in its solution, with various coefficients for each. Actually, all of the coefficients are "1" except for the range term (where it is "2") and the cosine of the fixation angle (where it is "$\sin\theta_R$"). Since the sine is never greater than one, we conclude that our estimate in spectral intensity is most sensitive to uncertainties in our measurement or calculation of range from target to sensor.

To go one step further ~ although the equation for $\frac{\delta(I_\lambda)}{I_\lambda}$ is most sensitive to uncertainties in the range, we point out that the quantity that is the least well known – the one with the most uncertainty – is probably the atmospheric transmission, τ_{ATM}. World-wide atmospheric conditions are not necessarily known, so considerable estimation and interpolation between weather stations be done to provide reasonable inputs to MODTRAN to generate an atmospheric transmission profile for any given collection location.