SUGGESTED SOLUTIONS (ODD)

CHAPTER 4

NOTE: Use three-digit precision for all calculations unless otherwise stated or implied.

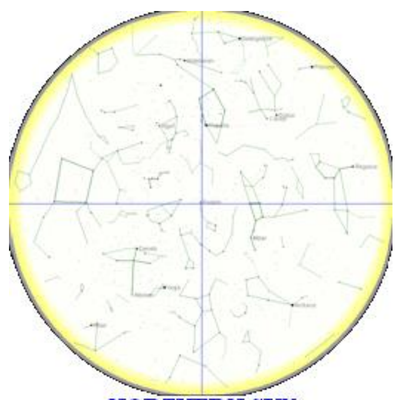
4-1. *Solid Angles.* Using both the exact and approximate formulas, calculate and compare …

(a) the solid angles, in steradians, subtended by symmetrical cones having (full) interior angles of 5º, 10°, 20°, 50°, 100°, and 180°.

(b) the (full) interior cone angles, in degrees, of symmetrical solid angle cones subtending π sr, π/2 sr, π/4 sr, π/10 sr, π/100 sr, and π/1000 sr.

☞ See spreadsheet Chapter 04 ~ Suggested Solutions (Odd).xlsx. ☜

4-3. *Probability of Seeing a Star in a Random Direction.* When we look up into the night sky, we note that there is only one visible star, Polaris, within ½° of the North Pole (in *any* direction), **but** there is NO star within a similar cone looking at the South Pole. (We determine this by consulting a star chart, or by asking someone who lives in the Southern Hemisphere.) Use these data to estimate the number of visible stars you should be able to see on a clear night (horizon to horizon).



NORTHERN SKY

SUGGESTED SOLUTION: Noting that only Polaris is within ½° of the North Pole suggests that if we looked into a 1° cone, centered on the North Pole, we would see **one** star. Similarly, looking into a 1° cone centered on the South Pole, we would see **zero** stars. Based on these limited observations, we speculate that the probability of seeing a star within a 1° cone in *any* random direction is 50%. The solid angle measure of the observation cone is

$$\Omega_{OBS} \approx \frac{\pi\,\alpha^2}{4} = \frac{\pi \times (1°)^2}{4}\left(\frac{\pi}{180°}\right)^2 \approx 2.39\times10^{-4}\ \text{sr}$$

SOUTHERN SKY

and there are $\dfrac{2\pi}{2.39\times10^{-4}} \approx 26{,}262$ such cones in a hemisphere. Taking this as the number of possible *random* directions we could look, and applying the 50% probability guesstimate, we calculate we should be able to see about 13,131 visible stars.

COMMENT: The Yale Bright Star Catalog lists 9110 stars brighter than visual magnitude 6.5, which is considered to be the usual visual limit (depending on where you are, how clear the sky is, how good your eyes are, etc.). Presumably half of these stars should be above your horizon at any given time (~ 4555 stars), so our simple-minded guess apparently overestimates by a factor of three or so. This still isn't bad considering the crudeness of our initial observations (namely just two data points).

4-5. *Collecting Power and Photons.* In the visible bandpass (0.4 – 0.7 μm), how much power is collected from a 100 W light bulb through a sensor's four inch diameter aperture at a distance of 10 feet? How much power is collected at a distance of one mile? At these two distances, how many photons are collected in one second of integration time?

SUGGESTED SOLUTION: Phenomenologically, the power collected through an aperture is ~

$$\Phi_{COLLECTED} = \begin{bmatrix} \text{Power per} \\ \text{unit area} \\ \text{at aperture} \end{bmatrix} \times \begin{bmatrix} \text{Aperture} \\ \text{area} \end{bmatrix} = \mathrm{E} A_R$$

where the irradiance, , is given radiometrically for a point source as $\mathrm{E} = \frac{I}{R^2}$. We only have to assume that our light bulb is an isotropic radiator: $I = \frac{\Phi_{EMITTED}}{4\pi}$. With $A_R = \frac{\pi D^2}{4}$, we have ~

$$\Phi_{COLLECTED} = \frac{\left(\frac{\Phi_{EMITTED}}{4\pi}\right)}{R^2}\left(\frac{\pi D^2}{4}\right) = \frac{\Phi_{EMITTED} D^2}{16 R^2}$$

(Notice that we have solved the problem "in letters," so we do not have to calculate intermediate results for different values of the variables. This is recommended procedure for all problems.) For data, we first refer back to Problem 2-17 where we found that a 100 W light bulb emits only 2.67 W in the visible band; then we convert D = 4 in ≈ 0.102 m, R = 10 ft ≈ 3.05 m, and R = 1 mi ≈ 1.61×10^3 m. Plugging in:

at 10 feet, $\Phi_{\mathrm{COLLECTED}} \approx 1.87 \times 10^{-4}$ W, and

at 1 mile, $\Phi_{\mathrm{COLLECTED}} \approx 6.70 \times 10^{-10}$ W.

Finding the number of photons collected has both a simple and a complicated answer. The simple answer is to assume that all of the photons are of one wavelength, say $\bar{\lambda} \approx 0.555$ μm, divide our power [energy per unit time] by the energy per photon, $E_{PHOTON} = hc/\bar{\lambda}$, and multiply by the integration time, Δt = 1 s ~

$$N_{COLLECTED} = \frac{\Phi_{EMITTED}\bar{\lambda} D^2 \Delta t}{16 hc R^2}$$

This gives us:

at 10 feet, $\mathrm{N}_{\mathrm{COLLECTED}} \approx 5.21 \times 10^{14}$ photons, and

at 1 mile, $\mathrm{N}_{\mathrm{COLLECTED}} \approx 1.87 \times 10^{9}$ photons.

The complicated answer is to consider how the light bulb emits photons *distributed* over the bandpass. Naturally, this is left as an exercise for the student. The introduction of emissivity of the light bulb filament would add a further complication.

4-7. *Seeing a Target against a Uniform Background.* An electro-optic sensor sees *both* a large, extended Lambertian target filling its field of view *and* a bright point source target on its boresight. Both sources provide the same irradiance on the sensor's aperture. If the sensor is moved to one half its original distance from the targets, how does the irradiance at its aperture change? How does the irradiance on its focal plane change?

SUGGESTED SOLUTION: In general, irradiance on aperture will be the sum of irradiances from all objects in a sensor's field of view. In this case, we initially have $\mathrm{E}_{EXTENDED} = \mathrm{E}_{POINT} \equiv \mathrm{E}_0$, so $\mathrm{E}_{INITIAL} = \mathrm{E}_{EXTENDED} + \mathrm{E}_{POINT} = 2\mathrm{E}_0$. If we believe the formula we have derived for irradiance at aperture from an extended source, $\mathrm{E}_{EXTENDED} \approx L\Omega$, then there is NO dependence on distance from the source. Therefore, it doesn't matter how far away we are from the source; the irradiance from the extended source will be the same, in particular when the distance is halved. For the point source, however, $\mathrm{E}_{POINT} = \dfrac{I}{R^2} \rightarrow \dfrac{I}{\left(R/2\right)^2} = 4\dfrac{I}{R^2} = 4\mathrm{E}_0$. Hence, moving to one-half the distance results in $\mathrm{E}_{HALF} = \mathrm{E}_0 + 4\mathrm{E}_0 = 5\mathrm{E}_0 = \dfrac{5}{2}\mathrm{E}_{INITIAL}$

Irradiance on the focal plane means the power per unit area falling on the focal plane to form an image. The image of an extended source is an extended image, but the image of a point source is a concentrated "point" image. (Point source imaging is discussed in Chapters 6 and 8.) When halving the distance to the targets, irradiance at aperture remains the same for the extended source, thus the irradiance of the extended image will not change. But the irradiance of the point image will increase proportional to the point source's irradiance on aperture. That is, the extended image "brightness" will not change, but the point source image brightness will increase fourfold.

4-9. *Detecting a Laser Designator.* Suppose an aircraft flying at 80,000 ft uniformly irradiates a circular spot on the ground 100 m in diameter with a 1000 W Nd:YAG laser (1.06 μm). Assume the terrain is a perfectly diffuse reflector with ρ = 0.25.

A. To detect the presence of the aircraft, we place an upward-looking sensor on the ground. What is the irradiance on its aperture from the laser when the aircraft is directly overhead?

B. What is the reflected radiance of the ground from the incident laser light?

C. If a <u>downward</u>-looking sensor (on a "sky hook") is 100 m above the center of the laser spot, what is the irradiance at its aperture? Assume the sensor is designed to only detect 1.06 μm laser light.

D. Which sensor (Part A or Part C) has a better chance of detecting the plane, and why?

E. Now suppose that the center of the spot is 500 m away (horizontally) from the downward-looking sensor – what is the irradiance on its aperture?

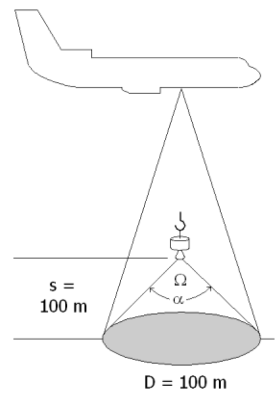
SUGGESTED SOLUTION:

A. The irradiance on aperture of an upward-looking sensor will be the same as the irradiance of the laser on the ground …

$$E_{GROUND} \approx \frac{\Phi_{1.06}}{\pi D^2 / 4} = \frac{(4)(1000\text{ W})}{(\pi)(100\text{ m})^2} \approx 0.127\,\frac{\text{W}}{\text{m}^2}$$

B. Assuming the ground is Lambertian (fully diffuse) …

$$L_{GROUND} = \frac{M_{GROUND}}{\pi} = \frac{\rho_{GROUND} E_{GROUND}}{\pi} = \frac{(0.25)(0.127\text{ W}\cdot\text{m}^2)}{\pi} \approx 0.0101\,\frac{\text{W}}{\text{m}^2\text{sr}}$$

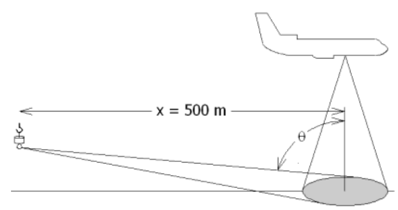


C. This is not quite so simple because the sensor's Field Of View (FOV) angle $\alpha = 2\tan^{-1}\frac{50\text{ m}}{100\text{ m}} \approx 53.1°$ is not "small" (see figure at right).[1] We should use the full-up formula for calculating solid angle rather than an approximation …

$$E_{SENSOR} = L\Omega = 2\pi L\left(1-\cos\frac{\alpha}{2}\right)$$

$$= (2\pi)\left(0.0101\,\frac{\text{W}}{\text{m}^2\text{sr}}\right)\left(1-\cos\frac{53.1°}{2}\right) \approx 6.70\times10^{-3}\,\frac{\text{W}}{\text{m}^2}$$

D. Well duh! The irradiance on the upward-pointing sensor is about one and a half times greater than that on the downward-pointing sensor. **BUT** the upward-pointing sensor will only see the airplane when it is directly overhead (well, within 50 m of overhead). Presumably the downward-pointing sensor is designed to have a Field of Regard (FOR) from horizon to horizon (like a fisheye lens), but also a filter so that the only thing it "sees" is the laser light. If your mission is to detect enemy aircraft flying around lasing you, which sensor would you want to detect it (you don't necessarily know where the aircraft is going to be flying)?



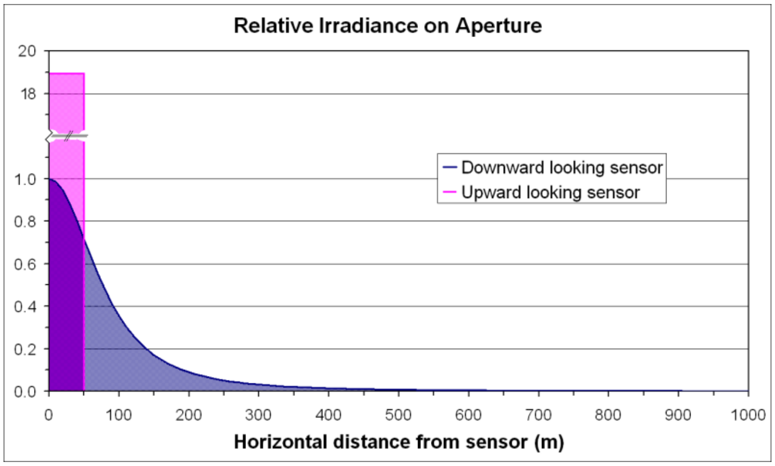
E. This is a little trickier, but we assume that the only input to the sensor is from the "extended source" of the laser spot on the ground, filling the Field of View (FOV) of a solid angle subtended by that spot. Since we have a fairly oblique view of the ground from the sensor, we'll go ahead and approximate the solid angle …

$$E_{SENSOR} \approx L\Omega \approx L\frac{A_\perp}{R^2} = L\frac{A\cos\theta}{R^2} = L\frac{\frac{\pi D^2}{4}\frac{s}{R}}{R^2} = \frac{L\pi D^2 s}{4R^3} = \frac{L\pi D^2 s}{4(x^2+s^2)^{3/2}}$$

$$= \frac{(0.0101\text{W}\cdot\text{m}^{-2}\text{sr}^{-1})(\pi)(100\text{m})^2(100\text{m})}{(4)[(500\text{m})^2+(100\text{m})^2]^{3/2}} \approx 5.98\times10^{-5}\,\frac{\text{W}}{\text{m}^2}$$

[1] On the other hand, the calculations we made for 4-1 showed that the solid angle approximation for α = 53° is in error by only about 1.8%.

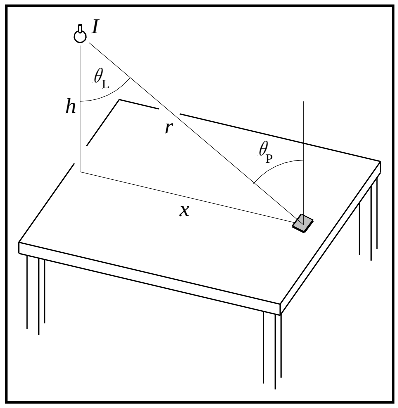
Note that in the limit of $x \to 0$, our formula is $E_{SENSOR} \approx L \frac{\pi D^2}{4s^2}$ where $\Omega \approx \frac{\pi D^2}{4s^2} \approx \frac{\pi \alpha^2}{4}$ is the approximation for the sky-hooked sensor's FOV we could have used in Part C. The following plot shows the relative irradiance as a function of the horizontal distance from the sensors to the center of the aircraft's laser spot.



4-11. *Irradiance from a Point Source.* Refer to the sketch at right. A light bulb with intensity I is suspended above a table, and a business card lies on the table. Consider the size of the card to be small with respect to all other dimensions. Note that $\theta_L = \theta_P$.

A. For a fixed height of the bulb, h, find the distance, x, for maximum irradiance on the card.

B. For a fixed distance, x, find the height of the bulb, h, that maximizes irradiance on the card.



SUGGESTED SOLUTION:

A. The only tricky thing about this problem is recognizing that the card lying on the table is "looking up." That is, the light bulb is at a fixation angle θ_P with respect to the card. In terms of the dimensions given in the sketch ~

$$E = \frac{I}{R^2}\cos\theta_R = \frac{I}{r^2}\cos\theta_P, \text{ where } r^2 = x^2 + h^2 \text{ and } \cos\theta_P = \cos\theta_L = \frac{h}{r} = \frac{h}{\sqrt{x^2+h^2}}$$

Then we have ~

$$E = \frac{I}{(x^2+h^2)}\frac{h}{\sqrt{x^2+h^2}} = \frac{I\,h}{(x^2+h^2)^{3/2}} \qquad [1]$$

<u>By inspection</u>, we note that x only appears in the denominator. The irradiance will thus be maximum when the denominator is minimum, and that occurs for $x \to 0$:

$$E_{MAX} = \frac{I\,h}{(0^2+h^2)^{3/2}} = \frac{I\,h}{h^3} = \frac{I}{h^2}$$

B. This requires finding the maximum value of equation [1] in Part A. There are a couple of ways to do this. If you don't know calculus, then the graphical solution we show in the spreadsheet will get you an approximate answer. If you do know calculus, then the solution is the time-honored method of taking the derivative and setting it equal to zero ~

$$\frac{d}{dx}\mathrm{E} = I\left\{\frac{1}{(x^2+h^2)^{3/2}} + \frac{-3/2\, h\cdot 2h}{(x^2+h^2)^{5/2}}\right\} \overset{\text{set}}{=\!=} 0$$

$$\frac{1}{(x^2+h^2)^{3/2}} - \frac{3h^2}{(x^2+h^2)^{5/2}} = 0$$

$$1 - \frac{3h^2}{x^2+h^2} = 0$$

$$x^2 + h^2 - 3h^2 = 0$$

$$h = x/\sqrt{2}$$

4-13. ***Irradiance from a Satellite.*** A small spherical satellite (1 m radius) in a 350 km circular orbit has an estimated steady state temperature of 290 K when in the Earth's shadow. Assuming its surfaces are perfectly diffuse and that its average reflectivity is 0.9 in the IR (see Problem 3-5), calculate the number of thermally emitted photons per unit area per second received from the satellite at a ground-based sensor when the satellite is directly overhead. The detector operates in a narrow wavelength band ($\Delta\lambda$ = 1.00 μm), centered on the wavelength of maximum emission.

SUGGESTED SOLUTION: As a first cut, note that Wien's Displacement Law suggests that a 290 K blackbody (or graybody for that matter) has maximum emission at a wavelength of ~

$$\lambda_{MAX} \approx \frac{3000}{290} \approx 10.3\,\mu\text{m}\ \ \text{(rule of thumb)}\ \ \text{or}\ \approx \frac{2897.8}{290} \approx 9.99\,\mu\text{m}\ \ \text{(more precisely)}.$$

Since our value for the temperature of the satellite is only an estimate anyway (and the satellite's temperature is certainly not constant as it cools in the Earth's shadow), we'll just use 10 μm. Also noting that the bandpass is small $\left(\Delta\lambda/\lambda \approx 0.1\right)$, we can piece together some phenomenology to get ~

$$\left\{\frac{photons}{area\cdot time}\right\} = \frac{\left\{\frac{energy}{area\cdot time}\right\}}{\left\{\frac{energy}{photon}\right\}} = \frac{\left\{\frac{power}{area}\right\}}{\left\{\frac{energy}{photon}\right\}} = \frac{\mathrm{E}}{hc/\lambda} = \frac{I/R^2}{hc/\lambda} = \frac{\left(\frac{\Phi}{4\pi}\right)/R^2}{hc/\lambda} = \frac{\tilde{o}\overline{B}A/4\pi R^2}{hc/\lambda}$$

$$= \frac{\tilde{o}\overline{B_\lambda}\Delta\lambda\,\pi r^2/4\pi R^2}{hc/\lambda} = \frac{\tilde{o}\overline{B_\lambda}\Delta\lambda\, r^2\lambda}{4hcR^2} = \frac{(0.9)(26.5\frac{\text{W}}{\text{m}^2\mu\text{m}})(1.00\,\mu\text{m})(1\,\text{m})^2(10\times10^{-6}\,\text{m})}{(4)(6.63\times10^{-34}\,\text{J}\cdot\text{s})(3\times10^8\,\frac{\text{m}}{\text{s}})(3.5\times10^5\,\text{m})^2}$$

$$\approx 2.45\times10^9\,\frac{\text{photons}}{\text{m}^2\text{s}}$$

where our value for spectral blackbody exitance at λ = 10.0 μm comes from the spreadsheet Blackbody.xlsx, for example, with the temperature set to 290 K.

If we learned our lesson from Problem 2-9, however, we know that there could be more to this problem, particularly if the bandpass was wider. Since photon energy is a function of wavelength, the proper formulation of this problem should be ~

$$\frac{photons}{area \cdot time} \approx \frac{r^2}{4hcR^2} \int_{BANDPASS} \varepsilon(\lambda) B_\lambda(\lambda, T) \lambda d\lambda$$

where we threw in some wavelength dependence in the emissivity for good measure.

4-15. *Comparing Emission and Reflection.* Assume the sun is a blackbody at 5900 K. Assume the Sahara Desert is a graybody with a steady state temperature of 315 K and emissivity $\bar{\varepsilon} \approx 0.914$. Calculate the wavelength at which reflected and emitted radiance are the same. (Ignore atmospheric attenuation.)

SUGGESTED SOLUTION: (Note that since we want to calculate radiances at specific wavelengths, we really mean *spectral* radiances in this problem.) First a little theory ~

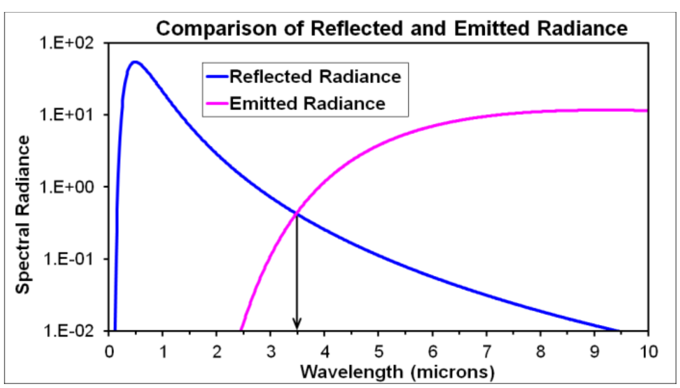
$$L_{\lambda,REFLECTED} = \frac{M_{\lambda,REFLECTED}}{\pi} = \frac{\rho \mathrm{E}_{\lambda,INCIDENT}}{\pi} = \frac{(1-\varepsilon)\dfrac{I_{\lambda,SUN}}{R^2}}{\pi} = \frac{(1-\varepsilon)\dfrac{\Phi_{\lambda,SUN}}{4\pi}}{\pi R^2}$$

$$= \frac{(1\text{-}\varepsilon)\mathrm{B}_{\lambda,SUN} A_{SUN}}{4\pi^2 R^2} = \frac{(1-\varepsilon) B_{\lambda,SUN} 4\pi r_{SUN}^2}{4\pi^2 R^2} = \frac{1-\varepsilon}{\pi} \frac{r_{SUN}^2}{R^2} B_{\lambda,SUN}$$

and ~

$$L_{\lambda,EMITTED} = \frac{M_{\lambda,EMITTED}}{\pi} = \frac{\varepsilon B_{\lambda,EMITTED}}{\pi}.$$

Then we calculate the radiances on a spreadsheet and plot them ~



The plot suggests that the cross-over is around 3.5 μm, and a closer inspection of the spreadsheet reveals that it is actually between 3.47 and 3.48 μm.

4-17. *Estimating Temperature.* If the actual total solar irradiance on the Earth (called "insolence") is 1375 W·m^{-2}, then calculate the *effective* temperature of the Sun.

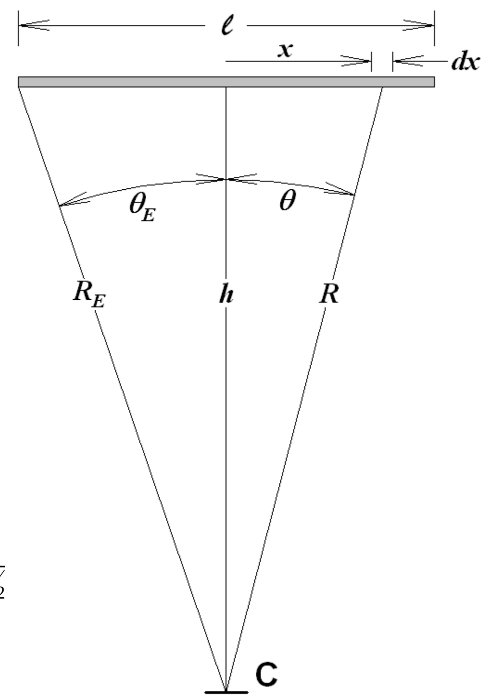
SUGGESTED SOLUTION: Assuming the Sun is a blackbody and that we are talking here about its insolence at all wavelengths ~

$$\mathrm{E} = \frac{I_{SUN}}{R^2} = \frac{\Phi_{SUN}/4\pi}{R^2} = \frac{B_{SUN} A_{SUN}}{4\pi R^2} = \frac{\sigma_{SB} T_{SUN}^4 \, 4\pi r_{SUN}^2}{4\pi R^2} = \sigma_{SB} \frac{r_{SUN}^2}{R^2} T_{SUN}^4$$

$$T = \sqrt[4]{\frac{R^2}{r_{SUN}^2} \frac{\mathrm{E}}{\sigma_{SB}}} = \sqrt[4]{\frac{(1.5\times10^8\,\mathrm{km})^2 (1375\,\mathrm{W/m^2})}{(6.95\times10^5\,\mathrm{km})^2 (5.67\times10^{-8}\,\mathrm{W/(m^2K^4)})}} \approx 5797\,\mathrm{K}$$

4-19. *Line Source.* A common fluorescent light bulb is four feet long and radiates 40 W of light in the visible. What is the irradiance on a business card lying on a table six feet directly under the middle of the lamp?

SUGGESTED SOLUTION: Let the light bulb be of length λ, then define the "power per unit length" emitted by the tube as $\Phi_\ell = \dfrac{d\Phi}{dx}$ such that $\int_0^\ell \Phi_\ell \, dx = \Phi$. (Note the symbolic allusion to a distribution function, just like the spectral radiometric quantities.) Then the intensity of an element of the bulb of length dx is $dI = \dfrac{d\Phi}{4\pi} = \dfrac{\Phi_\ell \, dx}{4\pi}$, supposing that every element of the tube can radiate in all directions – ignoring the end directions being obscured by the physical size of the tube itself.



The element of irradiance on card **C** due to element of intensity dI is then ~

$$d\mathrm{E} = \frac{dI}{R^2}\cos\theta = \frac{\Phi_\ell \, dx}{4\pi} \frac{1}{(x^2+h^2)} \frac{h}{\sqrt{x^2+h^2}} = \frac{\Phi_\ell \, h}{4\pi} \frac{dx}{(x^2+h^2)^{3/2}}$$

where the geometry is as defined in the sketch, and we took a clue from the previous "card on the table" Problem 4-11. To find the total irradiance on the card, all we have to do is sum up (integrate) the contributions from all of the intensity elements of the light bulb from end to end ~

$$\mathrm{E} = \int d\mathrm{E} = \frac{\Phi_\ell h}{4\pi} \int_{-\ell/2}^{+\ell/2} \frac{dx}{\left(x^2+h^2\right)^{3/2}} = 2\frac{\Phi_\ell h}{4\pi} \int_{0}^{\ell/2} \frac{dx}{\left(x^2+h^2\right)^{3/2}} = \frac{\Phi_\ell h}{2\pi} \left[\frac{x}{h^2\sqrt{x^2+h^2}} \right]_0^{\ell/2}$$

$$= \frac{\Phi_\ell}{2\pi h} \left[\frac{\ell/2}{\sqrt{\left(\ell/2\right)^2+h^2}} \right] = \frac{\Phi_\ell}{2\pi h} \left[\frac{\ell/2}{R_E} \right] = \frac{\Phi_\ell}{2\pi h} \sin\theta_E$$

where R_E and θ_E are the distance and angle to the end of the bulb, respectively.

Finally, sticking in some numbers ~

$$\ell = 4\text{ ft} = 1.22\text{ m},\;\; h = 6\text{ f}t = 1.83\text{ m},\;\; R_E = \sqrt{\left(2\text{ ft}\right)^2 + \left(6\text{ ft}\right)^2} = 6.32\text{ ft} = 1.93\text{ m}$$

$$\Phi_\ell = \frac{40\text{ W}}{1.22\text{ m}} = 32.8\text{ W/m},\;\; \sin\theta_E = \frac{1.22\text{ m}/2}{1.93\text{ m}} = 0.316$$

$$\mathrm{E} = \frac{\left(32.8\text{ W/m}\right)\left(0.316\right)}{\left(2\pi\right)\left(1.83\text{ m}\right)} \approx 0.901\text{ W/m}^2$$