SUGGESTED SOLUTIONS (ODD)

CHAPTER 11

NOTE: Use three-digit precision for all calculations unless otherwise stated or implied. You will discover that more precision is definitely needed for a couple of these problems.

11-1. *Using Kepler's Laws*. An Earth satellite is observed to have a height of perigee of 200 NM and a height of apogee of 800 NM. Find the semi-major axis, semi-minor axis, eccentricity, period, specific angular momentum, and specific energy of the orbit. [Recall that 5400 NM = 10,000 km by definition]

SUGGESTED SOLUTION: A tricky thing to this problem is converting the nautical miles (NM) to kilometers. (Since the US Navy was engaged in our first satellite programs, it is not uncommon to see orbital parameters expressed in NM.) The conversion goes like this, adding the mean radius of the Earth to get apogee and perigee distances ~

$$r_a = h_a + R_E = (800\,\text{NM})\frac{10{,}000\,\text{km}}{5400\,\text{NM}} + 6366\,\text{km} \approx 7847\,\text{km and}$$

$$r_p = h_p + R_E = (200\,\text{NM})\frac{10{,}000\,\text{km}}{5400\,\text{NM}} + 6366\,\text{km} \approx 6736\,\text{km}.$$

Now we can find the semi-major axis as the arithmetic mean of r_a and r_p:

$$a = \frac{r_a + r_p}{2} = \frac{7847\text{ km} + 6736\text{ km}}{2} \approx 7292\text{ km}.$$

Before finding the semi-minor axis, we solve for the eccentricity as ~

$$\hat{e} = \frac{c}{a} = \frac{r_a - r_p}{r_a + r_p} \approx \frac{7847\,\text{km} - 6736\,\text{km}}{7847\,\text{km} + 6736\,\text{km}} \approx 0.0762 \quad \text{(almost circular).}$$

From the eccentricity, we could calculate the focal distance, $c = a\varepsilon$, and plug it in to find the semi-minor axis from

$$b^2 + c^2 = a^2 \quad \Rightarrow \quad b = \sqrt{a^2 - c^2}\,,$$

but instead we'll make the substitution (some algebra left to the student):

$$b = \sqrt{a^2 - a^2\hat{e}^2} = a\sqrt{1 - \hat{e}^2} = \frac{r_a + r_p}{2}\sqrt{1 - \frac{\left(r_a - r_p\right)^2}{\left(r_a + r_p\right)^2}} = \sqrt{r_a r_p}\,,$$

which is the geometric mean of r_a and r_p. Thus,

$$b = \sqrt{\left(7847\text{ km}\right)\left(6736\text{ km}\right)} \approx 7270\text{ km}$$

Using Kepler's Third Law, we can easily calculate the satellite's period ~

$$P = \sqrt{\frac{4\pi^2}{G_N M_E} a^3} = \sqrt{\frac{4\pi^2}{\mu_E} a^3} \approx \sqrt{(9.895\times10^{-5}\text{s}^2\text{km}^{-3})(7292\,\text{km})^3} \approx 6194\,\text{s} \approx 103^{\text{m}}\,14^{\text{s}} \approx 1^{\text{h}}\,23^{\text{m}}\,14^{\text{s}}.$$

Next we will calculate the specific energy as ~

$$E_{sp} = -\frac{\mu_E}{2a} \approx -\frac{\left(3.99\times10^5\,\text{km}^3\text{s}^{-2}\right)}{(2)(7292\text{ km})}\left[\frac{10^3\,\text{m}}{1\,\text{km}}\right]^2\left[\frac{1\,\text{kg}}{1\,\text{kg}}\right] \approx -2.74\times10^7\,\text{J}\cdot\text{kg}^{-1}$$

where the conversion factors are necessary to make the units work out right. From the energy, we can calculate the satellite's speed at apogee (and perigee):

$$E_{sp} = \frac{v^2}{2} - \frac{\mu_E}{r} \quad\Rightarrow\quad v = \sqrt{2\left(E_{sp} + \frac{\mu_E}{r}\right)}$$

$$v_a = \sqrt{2\left(E_{sp} + \frac{\mu_E}{r_a}\right)} = \sqrt{2\left(-2.74\times10^7\,\frac{\text{J}}{\text{kg}}\left[\frac{1\,\text{km}}{10^3\,\text{m}}\right]^2 + \frac{3.99\times10^5\,\text{km}^3\text{s}^{-2}}{7847\text{ km}}\right)} \approx 6.85\,\text{km}\cdot\text{s}^{-1}$$

$$v_p = \sqrt{2\left(E_{sp} + \frac{\mu_E}{r_p}\right)} = \sqrt{2\left(-2.74\times10^7\,\frac{\text{J}}{\text{kg}}\left[\frac{1\,\text{km}}{10^3\,\text{m}}\right]^2 + \frac{3.99\times10^5\,\text{km}^3\text{s}^{-2}}{6736\text{ km}}\right)} \approx 7.98\,\text{km}\cdot\text{s}^{-1}.$$

Then its specific angular momentum is:

$$\ell_{sp} = r_a v_a = (7847\text{ km})\left(6.85\text{ km}\cdot\text{s}^{-1}\right) \approx 5.38\times10^4\,\text{km}^2\text{s}^{-1} \quad \text{or}$$
$$\ell_{sp} = r_p v_p = (6736\text{ km})\left(7.98\text{ km}\cdot\text{s}^{-1}\right) \approx 5.38\times10^4\,\text{km}^2\text{s}^{-1}.$$

11-3. ***Using Ellipse Dimensions*.** For a certain Earth satellite it is known that its semi-major axis is $a \approx 30 \times 10^6$ ft, and its orbital eccentricity is $\hat{e} \approx 0.2$. Find its perigee and apogee distances, r_p and r_a, from the center of the Earth, and its period, specific energy, and specific angular momentum. Also, calculate its distance from the center of the Earth when its true anomaly is $\theta_P = 45°$, 90°, and 135°.

SUGGESTED SOLUTION: (Apologies for giving a distance in feet, but that was not uncommon in the Mercury/Gemini/Apollo era.) Convert the 30 million feet into 9144 km, and press on with the calculation. First, the apogee and perigee distances are

$$r_a = a(1+\hat{e}) \approx (9144\text{ km})(1+0.2) \approx 10{,}970\,\text{km}$$
$$r_p = a(1-\hat{e}) \approx (9144\text{ km})(1-0.2) \approx 7315\,\text{km}.$$

The period is straightforwardly calculated:

$$P = \sqrt{\frac{4\pi^2}{G_N M_E} a^3} = \sqrt{\frac{4\pi^2}{\mu_E} a^3} \approx \sqrt{(9.895\times10^{-5}\text{s}^2\text{km}^{-3})(9144\,\text{km})^3} \approx 8698\,\text{s} \approx 144^{\text{m}}\,58^{\text{s}} \approx 2^{\text{h}}\,24^{\text{m}}\,58^{\text{s}}.$$

The specific energy is:

$$E_{sp} = -\frac{\mu_E}{2a} \approx -\frac{\left(3.99\times10^5\,\text{km}^3\text{s}^{-2}\right)}{(2)(9144\text{ km})}\left[\frac{10^3\text{m}}{1\text{km}}\right]^2\left[\frac{1\text{kg}}{1\text{kg}}\right] \approx -2.18\times10^7\,\text{J}\cdot\text{kg}^{-1}$$

where the conversion factors are necessary to make the units work out right. From the energy, we can calculate the satellite's speed at apogee (and perigee):

$$E_{sp} = \frac{v^2}{2} - \frac{\mu_E}{r} \quad \Rightarrow \quad v = \sqrt{2\left(E_{sp} + \frac{\mu_E}{r}\right)}$$

$$v_a = \sqrt{2\left(E_{sp} + \frac{\mu_E}{r_a}\right)} = \sqrt{2\left(-2.18\times10^7\,\frac{\text{J}}{\text{kg}}\left[\frac{1\text{km}}{10^3\text{m}}\right]^2 + \frac{3.99\times10^5\,\text{km}^3\text{s}^{-2}}{10{,}970\text{ km}}\right)} \approx 5.40\,\text{km}\cdot\text{s}^{-1}$$

$$v_p = \sqrt{2\left(E_{sp} + \frac{\mu_E}{r_p}\right)} = \sqrt{2\left(-2.18\times10^7\,\frac{\text{J}}{\text{kg}}\left[\frac{1\text{km}}{10^3\text{m}}\right]^2 + \frac{3.99\times10^5\,\text{km}^3\text{s}^{-2}}{7315\text{ km}}\right)} \approx 8.09\,\text{km}\cdot\text{s}^{-1}.$$

Then its specific angular momentum is:

$$\ell_{sp} = r_a v_a = (10{,}970\text{ km})\left(5.40\text{ km}\cdot\text{s}^{-1}\right) \approx 5.92\times10^4\,\text{km}^2\text{s}^{-1} \quad \text{or}$$

$$\ell_{sp} = r_p v_p = (7315\text{ km})\left(8.09\text{ km}\cdot\text{s}^{-1}\right) \approx 5.92\times10^4\,\text{km}^2\text{s}^{-1}.$$

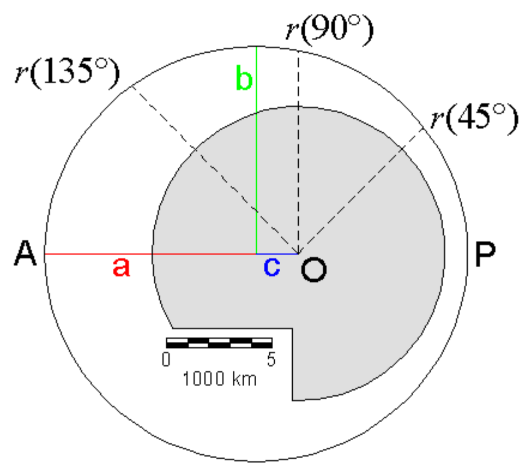
Lastly, the satellite's distance from the center of the Earth at various anomalies is found by ~

$$r(\theta_P) = \frac{a\left(1-\hat{e}^2\right)}{1+\hat{e}\cos\theta_P} \quad \Rightarrow$$

$$r(45°) = \frac{(9144\text{ km})\left(1-0.2^2\right)}{1+(0.2)\cos 45°} \approx 7691\text{ km},$$

$$r(90°) = \frac{(9144\text{ km})\left(1-0.2^2\right)}{1+(0.2)\cos 90°} \approx 8778\text{ km, and}$$

$$r(135°) = \frac{(9144\text{ km})\left(1-0.2^2\right)}{1+(0.2)\cos 135°} \approx 10{,}224\text{ km.}$$



The sketch on the right shows the satellite orbit to scale, compared to the Earth. Incidentally, the position of the satellite at 90° marks what is known as the "latus rectum" or "line at right angles (to the major axis through a focus)" of an ellipse. It is usually designated as p, and is given by $p = a\left(1-\hat{e}^2\right) = a\left(1-\hat{e}\right)\left(1+\hat{e}\right)$.

11-5. ***A Kinematics Problem.*** A sounding rocket is fired vertically from White Sands Missile Range (WSMR). It achieves a burnout speed of $v_{BO} \approx 10{,}000$ ft/sec at an altitude above sea level of $z_{BO} \approx 100{,}000$ ft. Neglecting atmospheric drag, determine the maximum altitude the missile attains.

SUGGESTED SOLUTION: Remember the days of freshman physics when you solved motion problems with kinematics equations? Back then we used one-dimensional equations ~

$$a = -g, \quad v = v_0 + a(t - t_0), \quad z = z_0 + v_0(t - t_0) + \tfrac{1}{2}a(t - t_0)^2, \quad \text{and} \quad v^2 - v_0^2 = 2a(z - z_0)\,.$$

The last equation (without time in it) would seem to work here where we identify
$v_0 = 10{,}000\ \text{ft}/\text{sec} = 3.048\ \text{km s}^{-1}$, $z_0 = 100{,}000\ \text{ft} = 30.48\ \text{km}$, and $a = -9.81 \times 10^{-3}\ \text{km s}^{-2}$.
(Recall the acceleration is negative because altitude, *z*, is positive upward.) We then set *v* = 0 at altitude *z* when the rocket reaches the top of its trajectory, and solve:

$$z = z_0 - \frac{v_0^2}{2a} = 30.48\,\text{km} - \frac{(3.048\ \text{km s}^{-1})^2}{(2)(-9.81\times 10^{-3}\ \text{km s}^{-2})} \approx 504\ \text{km}.$$

The sharp-eyed of you will notice that the kinematics equation we used, which can be derived from $\dfrac{a}{v} = \dfrac{dv/dt}{dz/dt} = \dfrac{dv}{dz}$, is exactly the same as the conservation of energy equation we also used in freshman physics: $\tfrac{1}{2}mv^2 - mgz = \tfrac{1}{2}mv_0^2 - mgz_0$ where we have multiplied throughout by mass. That is, the sum of mechanical energies – motion (kinetic) and position (potential) – is constant in the absence of dissipation. We have that situation here (because we are ignoring air drag), BUT it is likely that **the acceleration is not constant**. Recall that from the gravitational force $F_G = ma = \dfrac{G_N M_E m}{r^2} = \dfrac{m\mu_E}{r^2}$ we have $a = \dfrac{\mu_E}{r^2}$ where $r = R_E + z$ and $\mu_E = G_N M_E$. Since our preliminary calculation suggests that $z/R_E \approx 504\ \text{km}/6370\ \text{km} \approx 8\%$ we can expect that our answer could be in error by about this amount.

To make a long story short, we need to use the correct form for the potential energy, $U = \int \vec{F}\cdot d\vec{r} = \int \dfrac{m\mu_E}{r^2}\,dr = -\dfrac{m\mu_E}{r}$ which gives us the <u>specific</u> energy equation we introduced:

$$\frac{v^2}{2} - \frac{\mu_E}{r} = \frac{v_0^2}{2} - \frac{\mu_E}{r_0} \quad \Rightarrow \quad -\frac{\mu_E}{R_E + z} = \frac{v_0^2}{2} - \frac{\mu_E}{R_E + z_0}\,.$$

Solving for altitude:

$$z = \frac{2\mu_E(R_E + z_0)}{2\mu_E - v_0^2(R_E + z_0)} - R_E$$

and substituting $\mu_E = (6.6732\times 10^{-20}\ \text{km}^3\text{s}^{-2})(5.979\times 10^{24}\ \text{kg}) \approx 3.9899\times 10^5\ \text{km}^3\text{s}^{-2}$ and $R_E \approx 6366.2\ \text{km}$ [that is, using a few more digits of precision] we get

$$z \approx \frac{(2)(3.9899\times 10^5\,\text{km}^3\text{s}^{-2})(6366.2\,\text{km} + 30.48\,\text{km})}{(2)(3.9899\times 10^5\,\text{km}^3\text{s}^{-2}) - (3.048\,\text{km s}^{-1})^2(6366.2\,\text{km} + 30.48\,\text{km})} - 6366.2\,\text{km} \approx 545\,\text{km}\,.$$

Sure enough, as advertised our answer is about 8% larger!

11-7. *Sidereal Timekeeping*. At noon (local mean time) on 21 June, a satellite in a highly elliptical orbit (HEO) is nearly overhead of Nenana, Alaska (64.5N, 149.1W). The satellite makes two revs per day on a repeating ground track. When the satellite is at exactly the same point in the sky (relative to the background stars) again on 21 September, 21 December, 21 March, and 21 June (the next year) approximately what [local mean] time is it in Nenana?

SUGGESTED SOLUTION: Since a sidereal day ($23^h\ 56^m\ 4.090524^s$) is $3^m\ 55.909476^s$ (235.909476 s) shorter than a solar day, the satellite appears in the same place in the sky (relative to the star background) approximately four minutes EARLIER each day.

From 21 June to 21 September is 92 days, so the satellite will return roughly 368 minutes before noon, or at about 6 AM. To get this answer a little closer, multiply: 235.909476 s$\times$ 92 $\approx$ 21,703.671792 s $\approx$ $06^h\ 01^m\ 43.671792^s$ earlier, which is 05:58:16.328208 in the morning.

From 21 September to 21 December is 91 days, so the satellite will return in roughly another 364 minutes earlier, or about midnight. To get this answer a little closer, multiply: 235.909476 s$\times$ 91 $\approx$ 21,467.762316 s $\approx$ $05^h\ 57^m\ 47.762316^s$ earlier, which is 00:00:28.565892, or only about a half minute past midnight.

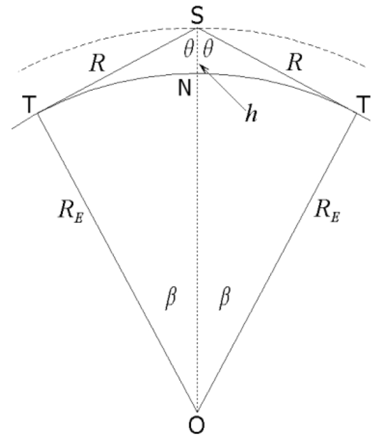
From 21 December to 21 March (assuming a non-leap year) is 90 (calendar) days, HOWEVER, notice the following. Two satellite revs later, the clock in Nenana has advanced one sidereal day, **but** the local mean time is 23:56:32.656416, or about three and a half minutes before midnight. <u>There are two instances of the satellite being in the same place in the sky on 21 December!!!</u> Therefore, on the 90th calendar day (21 March), the satellite will have passed through 91 sidereal days, and the local mean time in Nenana is about 6 PM. To get this answer a little closer, multiply: 235.909476 s$\times$ **91** $\approx$ 21,467.762316 s $\approx$ $05^h\ 57^m\ 47.762316^s$ earlier, which is 00:00:28.565892, or 18:02:40.803576.

From 21 March to 21 June is 92 days so the satellite will return in roughly another 368 minutes earlier, or about noon again. To get this answer a little closer, multiply: 235.909476 s$\times$ 92 $\approx$ 21,703.671792 s $\approx$ $06^h\ 01^m\ 43.671792^s$ earlier, which is 12:00:57.121784.

COMMENT: It might occur to you to ask why this is about a minute off from returning at noon where it started a year ago. The difference is in the leap year: the Earth makes about 365.25 revolutions in a year whereas the satellite passes through about 366.25 sidereal days. In approximately four years, the Earth will have rotated one extra time, and the satellite will have passed through an additional five sidereal days, so in four years, the satellite will again return to overhead at just about noon on the 21st of June.

11-9. *Revisit*. What are the field of regard (FOR) and orbital period, *P*, of a sensor on a satellite in low earth, circular orbit at 830 km altitude? If the inclination of the orbit is 0º, should the sensor be able to see Quito, Ecuador (latitude ≈ 0º) on two successive passes? (What other condition on the sensor is necessary? HINT: What is it's FOV, and how is it oriented?) If the inclination of the satellite's orbit is 60º, will the sensor be able to see Quito, Ecuador on two successive passes? [HINTS: The second part of the question may seem daunting, but consider the worst case inclination (90°). Then you will need to calculate the rotational speed of the Earth as it spins about its axis. Also, you need to remind yourself of the other condition on the sensor's FOV that is necessary.]

SUGGESTED SOLUTION: First, see the scale drawing which is a cross-section view: the satellite is at S and the center of the Earth is at O. The satellite's FOR extends to the horizon all around at tangent point T. Since we know $R_E \approx 6370\text{ km}$ and $\overline{\text{OS}} = R_E + h \approx 7200\text{ km}$, we can solve for R, β, and θ in triangle OTS:



$$R = \sqrt{\left(R_E + h\right)^2 - R_E^2} = \sqrt{7200^2 - 6370^2} \approx 3360\text{ km},$$

$$\beta = \cos^{-1}\left(\frac{R_E}{R_E + h}\right) = \cos^{-1}\left(\frac{6370}{7200}\right) \approx 27.8°\text{, and}$$

$$\theta = 90° - \beta = 90° - 27.8° \approx 62.2°.$$

There are various ways that we can now express field of regard. (1) In angular measure, it is not uncommon to see FOR given as $2\theta \approx 124.4°$. (2) In linear measure, the ground distance from horizon to horizon could be quoted as $2NT = 2R_E\beta = 2\left(6370\text{ km}\right)\left(27.8°\right)\left(\frac{\pi}{180°}\right) \approx 6180\text{ km}$. (Note that this is less than twice the slant range to the horizon.) (3) It is also acceptable to give the surface area under the satellite (where the formula is given without proof, but recall how to calculate a solid angle) as: $A = 2\pi R_E^2(1-\cos\beta) = 2\pi\left(6370\text{ km}\right)^2(1-\cos 27.8°) \approx 2.94\times10^7\text{ km}^2$.

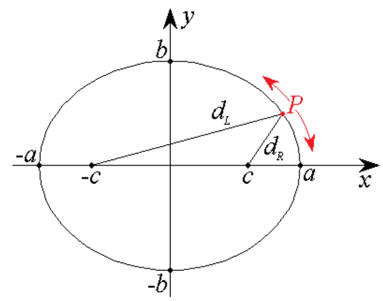
The period is a little easier to calculate from Kepler's third law:

$$P = \sqrt{\frac{4\pi^2}{\mu_E}a^3} = \sqrt{\frac{4\pi^2}{3.99\times10^5\text{ km}^3\text{s}^{-2}}(7200\text{ km})^2} \approx 6080\text{ s} \approx 1.69\text{ hr}$$

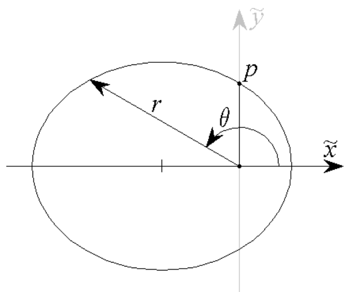
When this sensor is in an equatorial orbit (inclination = 0°) of course it could see a target located on the equator (Quito, Ecuador) on two successive passes. In fact it should be able to see Quito on *every* pass. But the problem stem hints that there is another condition that we need to consider here. It is simply that we must have that the sensor is capable of *pointing* at Quito. This will certainly be the case if the sensor is fixed nadir pointing or is a pushbroom type of configuration. Some whiskbroom type configurations may also have the correct properties to be able to point in the right direction at the right time.

For the next part of the problem, let's consider how far the Earth will rotate during one orbital period. This is the distance that the ground track of the LEO appears to drift to the West. Since the Earth makes one complete rotation (360°) in one *sidereal* day (23.9345 hours), we have that it rotates $\frac{360°}{23.9345\,hr}\times1.6881\,hr \approx 25.4°$ in that length of time. That is, our sensor will cross the equator approximately 25.4° further west on each pass. If we compare this to the fact that the FOR subtends $2\beta \approx 55.6°$ from the center of the Earth, then a little thought convinces us that any location on the equator should be visible from the satellite on at least two successive passes – some sites will be visible on three successive passes. The condition we have to watch out for, however, is again whether our sensor has sufficient freedom from its perch on its platform to swivel around and point at an intended target within its FOR.

11-11. ***Analytic Geometry.*** From the definition of an ellipse ~ namely that an ellipse is the locus of a point, P, in two-dimensional space such that the **sum** of the distances from the point to two other points, called *foci* (c and $-c$), is a constant ~



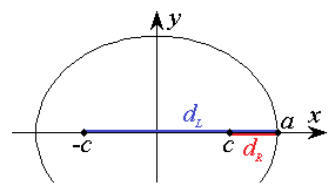
A. Derive the standard equation for an ellipse centered on the origin in the xy-plane: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where a and b are the semi-major and semi-minor axis dimensions, respectively (with the major axis on the x-axis and the minor axis on the y-axis); and



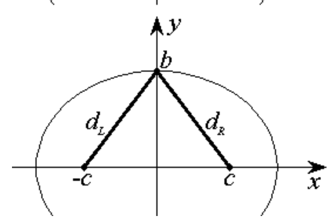
B. From the standard equation derive the polar form for the ellipse with the origin at the right-hand focus (c) and the angle measured counter-clockwise from the x-axis: $r=\frac{p}{1+\varepsilon\cos\theta}$, where p is the semi-latus rectum and $\varepsilon=\frac{c}{a}$ is the eccentricity.

SUGGESTED SOLUTION: There is some preliminary work to do before tackling the problem, working with the definition of an ellipse:

$$d_L+d_R=K\,,$$



where d_L and d_R are the distances to P from the left [(x,y) = $(-c,0)$] and right [(x,y) = $(c,0)$] foci, respectively, and K is the constant sum.



First, move P to the farthest-most point on the x-axis to the right [(x,y) = $(a,0)$]. Then, by inspection,

$$d_L=a+c \quad \text{and} \quad d_R=a-c\,,$$

so that

$$d_L+d_R=(a+c)+(a-c)=2a=K\,.$$

Thus the value of K is determined.

Second, move P to the top-most point on the y-axis [(x,y) = $(0,b)$], and from symmetry note that

$$d_L=d_R \quad \text{such that} \quad d_L+d_R=2d_L=2d_R=2a.$$

Thus with $d_L=d_R=a$ in this instance the Pythagorean Theorem tells us

$$a^2=b^2+c^2$$

since a is the hypotenuse of a right triangle whose legs are b and c.

A. Finding the standard equation of an ellipse centered on the origin is now just a matter of grinding through the algebra. Taking P to be any point on the ellipse as (x,y), proceed as follows:

$$d_L + d_R = 2a$$

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

$$(x+c)^2 + \cancel{y^2} = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + \cancel{y^2}$$

$$\cancel{x^2} + 2cx + \cancel{c^2} = 4a^2 - 4a\sqrt{(x-c)^2 + y^2} + \cancel{x^2} - 2cx + \cancel{c^2}$$

$$\cancel{4}cx - \cancel{4}a^2 = \cancel{4}a\sqrt{(x-c)^2 + y^2}$$

$$c^2x^2 - 2a^2cx + a^4 = a^2\left[(x-c)^2 + y^2\right]$$

$$c^2x^2 - \cancel{2a^2cx} + a^4 = a^2x^2 - \cancel{2a^2cx} + a^2c^2 + a^2y^2$$

$$\left(a^2 - c^2\right)x^2 + a^2y^2 = a^2\left(a^2 - c^2\right)$$

$$b^2x^2 + a^2y^2 = a^2b^2$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.}$$

Q.E.D.

B. To transform the standard equation into polar coordinates, first translate the origin to the right focus, $(x,y) = (c,0)$, to establish a parallel set of axes, $(\tilde{x}, \tilde{y})$:

$$\left.\begin{aligned} \tilde{x} &= x - c \\ \tilde{y} &= y \end{aligned}\right\} \quad \Leftrightarrow \quad \left\{\begin{aligned} x &= \tilde{x} + c \\ y &= \tilde{y} \end{aligned}\right. .$$

Thus the standard equation becomes

$$\frac{(\tilde{x}+c)^2}{a^2} + \frac{\tilde{y}^2}{b^2} = 1.$$

Before going farther, note at this point that when $\tilde{x} = 0$ we have

$$\frac{c^2}{a^2} + \frac{\tilde{y}^2}{b^2} = 1,$$

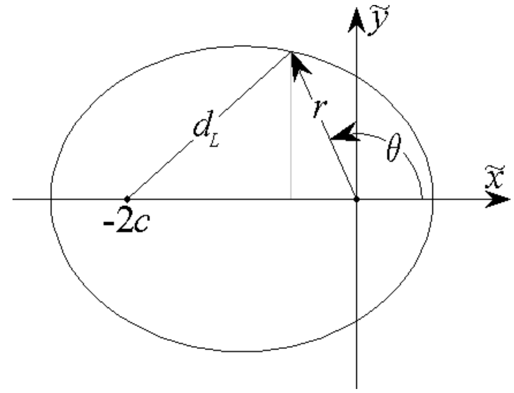
which leads us to

$$\begin{aligned} \tilde{y}^2 &= b^2\left(1 - \frac{c^2}{a^2}\right) = \left(a^2 - c^2\right)\left(1 - \frac{c^2}{a^2}\right) = a^2\left(\frac{a^2 - c^2}{a^2}\right)\left(1 - \frac{c^2}{a^2}\right) \\ &= a^2\left(1 - \frac{c^2}{a^2}\right)\left(1 - \frac{c^2}{a^2}\right) = a^2\left(1 - \varepsilon^2\right)^2 \equiv p^2 \end{aligned}$$

which is the semi-latus rectum, p, and where we have used the definition of eccentricity.

With that out of the way, now consider a transformation into polar coordinates:

$$\left.\begin{aligned} r^2 &= \tilde{x}^2 + \tilde{y}^2 \\ \tan\theta &= \frac{\tilde{y}}{\tilde{x}} \end{aligned}\right\} \Leftrightarrow \begin{cases} \tilde{x} = r\cos\theta \\ \tilde{y} = r\sin\theta \end{cases}$$



Referring to the figure at right, next calculate the sum of the distances from the foci to a point on the ellipse (that is, apply the definition again) where $d_R = r$ and d_L is found using Pythagorean Theorem again:

$$d_L + d_R = \sqrt{(r\sin\theta)^2 + (2c + r\cos\theta)^2} + r = 2a\,.$$

The rest is now just a matter of slugging through the algebra (and using a trig identity):

$$\sqrt{r^2\sin^2\theta + 4c^2 + 4cr\cos\theta + \mathrm{r}^2\cos^2\theta} = 2a - r$$

$$\sqrt{r^2\left(\sin^2\theta + \cos^2\theta\right) + 4c\left(c + r\cos\theta\right)} = 2a - r$$

$$\sqrt{r^2 + 4\left(c^2 + cr\cos\theta\right)} = 2a - r$$

$$\cancel{r^2} + \cancel{4}\left(c^2 + cr\cos\theta\right) = \cancel{4}a^2 - \cancel{4}ar + \cancel{r^2}$$

$$r\left(a + c\cos\theta\right) = a^2 - c^2$$

$$r = \frac{a^2 - c^2}{a + c\cos\theta} = \frac{a^2\left(1 - \frac{c^2}{a^2}\right)}{a\left(1 + \frac{c}{a}\cos\theta\right)} = \frac{a\left(1 - \varepsilon^2\right)}{1 + \varepsilon\cos\theta}$$

$$\boxed{r = \frac{p}{1 + \varepsilon\cos\theta}.} \qquad \text{Q.E.D.}$$

COMMENT: Since $p = a\left(1 - \varepsilon^2\right) = a\left(1 + \varepsilon\right)\left(1 - \varepsilon\right)$, we can easily show the perigee and apogee distances:

When $\theta = 0$: $r = \dfrac{a(1+\varepsilon)(1-\varepsilon)}{1+\varepsilon(+1)} = a(1-\varepsilon) \equiv r_p$, and

when $\theta = \pi$: $r = \dfrac{a(1+\varepsilon)(1-\varepsilon)}{1+\varepsilon(-1)} = a(1+\varepsilon) \equiv r_a$.