SUGGESTED SOLUTIONS (ODD)

CHAPTER 2

NOTE: Use three-digit precision for all calculations unless otherwise stated or implied.

2-1. ***Frequency, Energy, and Wavenumber.*** Some monochromatic beams of "light" have wavelengths of 555 nm, 2.50 μm, 12.0 μm, 1.00 cm, and 40.0 m in vacuum.

A. What are the frequencies of the electromagnetic waves of these radiations (in Hz)?

B. What are the energies of photons having these wavelengths (in J and eV)?

C. What are the wavenumbers associated with these wavelengths (in cm^{-1})?

☞ **See spreadsheet Chapter 02 ~ Suggested Solutions (Odd).xlsx for solution.** ☜

2-3. ***Number of Photons.*** A monochromatic source of radiation emits two watts of power at a wavelength of 1.50 μm.

A. How many photons is it emitting in one second?

B. How long would it take for this source to emit 10^{10} photons?

☞ **See spreadsheet Chapter 02 ~ Suggested Solutions (Odd).xlsx for solution.** ☜

2-5. ***Number of Photons.*** Suppose that equal *numbers* of red and blue photons of light are arriving at the Earth from the sun. What is the ratio of the energies of blue light to red light that is reaching us?

☞ **See spreadsheet Chapter 02 ~ Suggested Solutions (Odd).xlsx for solution.** ☜

2-7. ***Index of Refraction.*** What is the speed of light, and what is the wavelength of a "green" photon ($\lambda \approx 555$ nm) inside of …

(a) a diamond (index of refraction $n \approx 2.40$),

(b) a glass lens ($n \approx 1.60$), and

(c) a tank of water ($n \approx 1.33$)?

☞ **See spreadsheet Chapter 02 ~ Suggested Solutions (Odd).xlsx for solution.** ☜

2-9. *Photon Emission.* A certain object radiates 30.0 watts per square meter from its surface in a 1.00 μm bandpass centered on 10.0 μm. Estimate the number of photons emitted per second from one square meter of surface area.

SUGGESTED ANSWER: There are several levels of sophistication to this problem:

<u>(A) LEVEL ZERO</u>

Since we are given that an object emits $M = 30.0\ \text{W/m}^2$ from its surface, that's $M = 30.0$ joules per second per square meter. So for one square meter $\Phi = 30.0$ joules per second are emitted ($\Phi = M{\cdot}A$ where $A = 1.00\ \text{m}^2$). If we assume <u>all</u> of the photons have $\lambda = 10.0\ \mu\text{m}$, then the energy <u>per photon</u> is

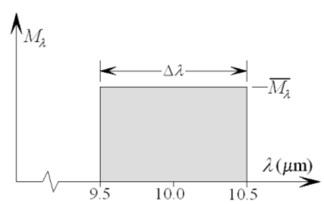
$$E = {hc}/{\lambda} = {(6.63\times10^{-34}\,\text{J}\cdot\text{s})(3.00\times10^{8}\,\text{m/s})}\Big/{(10\times10^{-6}\,\text{m})} \approx 1.99\times10^{-20}\,\text{J per photon}$$

The number of photons coming from our square meter per second is then

$$\dot{N} = {\Phi}/{E} = {30.0\,\text{J}\cdot\text{s}}\Big/{1.99\times10^{-20}\,\text{J/photon}} \approx 1.51\times10^{21}\,\text{photons per second.}$$

<u>(B) LEVEL ONE</u>

We also are told that the $M = 30\ \text{W/m}^2$ is in a $\Delta\lambda = 1\ \mu\text{m}$ band centered on $\lambda = 10\ \mu\text{m}$. So this is actually $M = \int_{9.5}^{10.5} M_\lambda d\lambda$ where M_λ is how the energy is *distributed* across the band. What is this distribution, and what does it mean in terms of photon emission? Answer: We don't know how it's distributed! But to make headway on this problem, we'll assume the simplest possible distribution of energy – a constant value for all wavelengths. When we remember that integral calculus (the expression above) is just calculating the area under a curve, then we can do the simple math for the area of the box sketched here:



$$30\,\frac{\text{W}}{\text{m}^2} = \int_{9.5}^{10.5} M_\lambda d\lambda = \text{height} \times \text{width} = \overline{M_\lambda} \times \Delta\lambda = \overline{M_\lambda} \times (1\,\mu\text{m})\,.$$

Thus

$$\overline{M_\lambda} = \frac{30\ \text{W/m}^2}{1\,\mu\text{m}} = 30\ \text{W/(m}^2\cdot\mu\text{m)}\,. \qquad \text{NOTE THE UNITS!}$$

Having chosen this simple, constant, value for *spectral* exitance, we see that there is as much power (energy per unit time) per unit area emitted at short wavelengths as at long wavelengths [speaking of the 9.5 – 10.5 μm band]. BUT we also know that <u>energy per photon</u> is inversely proportional to wavelength … so it takes fewer photons of shorter wavelength. How do we fold this into our calculation for the total number of photons emitted per second from our unit area?

This is how we do it: **spectrally**. We calculate the number of photons per second *per unit wavelength*, which is $\dot{N}_\lambda = {M_\lambda A}\big/{E} \quad \left[\dfrac{(\text{J/(s}\cdot\text{m}^2\mu\text{m}))(\text{m}^2)}{\text{J/photon}} = \dfrac{\text{photons}}{\text{s}\cdot\mu\text{m}} \right]$, and then <u>add them up</u> across the band. Adding them up means we INTEGRATE:

$$\dot{N} = \int \dot{N}_\lambda d\lambda = \int \frac{M_\lambda A}{hc/\lambda} d\lambda = \frac{\overline{M_\lambda} A}{hc} \int_{9.5}^{10.5} \lambda\, d\lambda = \frac{\overline{M_\lambda} A}{hc} \cdot \frac{\lambda^2}{2}\bigg|_{9.5}^{10.5}$$

{COMMENT: don't worry if analytical evaluation of integrals isn't in your bag of mathematical tricks. What you need to know is what an integral does – it finds (adds up) the area under a curve. We can always do it numerically if we must – see the next discussion for example.}

Evaluating this last expression requires some scrutiny of the units. Since we did the integral with wavelength in micrometers, attention to detail shows that we need to stick in a conversion factor to make it work out:

$$\dot{N} \approx \frac{(30\ \text{W/(m}^2\mu\text{m}))(1.00\ \text{m}^2)}{(2)(6.63\times10^{-34}\text{J}\cdot\text{s})(3.00\times10^{8}\text{m/s})}\left[(10.5\ \mu\text{m})^2 - (9.50\ \mu\text{m})^2\right]\left(\frac{1\ \text{m}}{10^6 \mu\text{m}}\right)$$
$$\approx 1.51\times10^{21}\text{photons/s}$$

Well! This isn't any different than our LEVEL ZERO answer, but only because we are working to three significant digits. We wouldn't expect it to be *much* different anyway because the bandpass is relatively narrow.

<u>(C) LEVEL TWO</u>

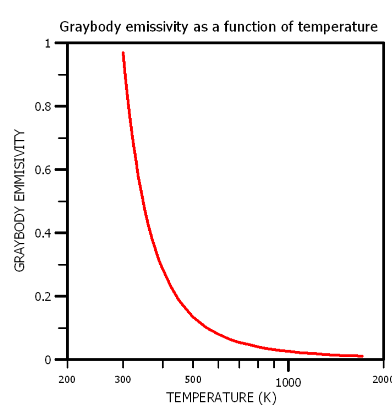
We just completed the calculation for a simple *assumed* energy distribution function. Since we don't know what the actual distribution function is, we could have used <u>anything</u> (the simpler the better!) and probably gotten nearly the same answer (unless we chose some really bizarre function). This time let's inject a little reality and assume that the energy distribution function is related to the Planck blackbody function.

This assumption introduces two new difficulties: what is the temperature, and what do we do about the emissivity? That is, we must satisfy the given requirement that

$$30.0\ \text{W/m}^2 = \int_{9.5}^{10.5} \grave{o}(\lambda) B_\lambda(\lambda,T)\, d\lambda$$

As usual, we make a simplifying assumption that across this narrow bandpass the emissivity is constant; it has the "graybody" value $\tilde{o}$:

$$30.0\ \text{W/m}^2 = \tilde{o}\int_{9.5}^{10.5} B_\lambda(\lambda,T)\, d\lambda$$



The relationship between allowed values of $\tilde{o}$ and temperature, *T*, is complicated by the integral, but is shown in the plot here. (Since the maximum value of $\tilde{o}$ is 1, the minimum temperature of this emitting object is ≈ 299 K.)

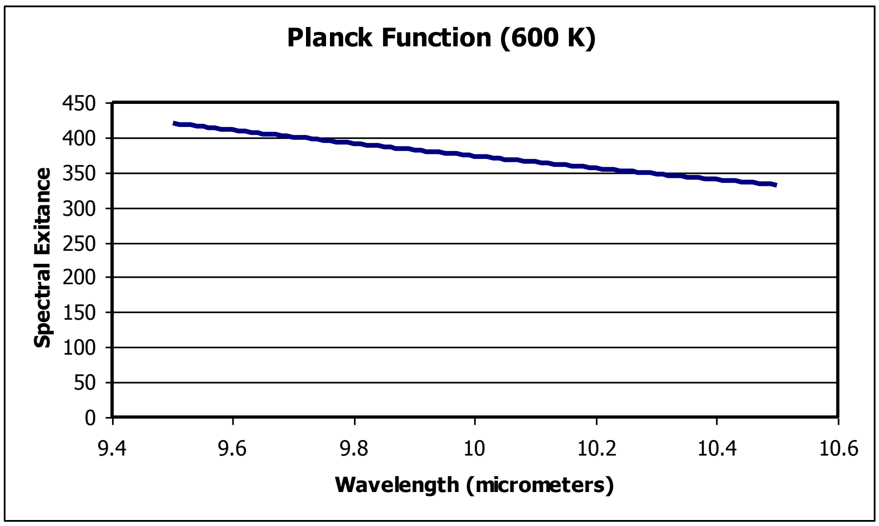
Let's suppose that the emitting object in this problem is the hot gas from the tailpipe of a jet engine. It is a very tenuous gas (you can see through it at most wavelengths) at about 600 K somewhere behind an aircraft, and its emissivity (from the plot) is ≈ 0.0802. Now we have that $M_\lambda = \bar{\varepsilon} B_\lambda(T)$ and we can proceed as before:

$$\dot{N} = \int \dot{N}_\lambda d\lambda = \int \frac{M_\lambda A}{hc/\lambda} d\lambda = \int \frac{\bar{\varepsilon} B_\lambda A}{hc/\lambda} d\lambda = \frac{\bar{\varepsilon} A}{hc}\int_{9.5}^{10.5} \lambda B_\lambda d\lambda$$

This time, the integral can only be done numerically and has a value of ≈ 3730 W·µm/m^2. (Note the units again!) We also have to pay attention to the units, as before, and stick in a conversion factor to get

$$\dot{N} = \frac{(0.0802)(1.00\,\mathrm{m}^2)(3730\,\mathrm{W}\cdot\mu\mathrm{m/m}^2)}{(6.63\times10^{-34}\,\mathrm{J}\cdot\mathrm{s})(3.00\times10^{8}\,\mathrm{m/s})}\left(\frac{1\,\mathrm{m}}{10^6\,\mu\mathrm{m}}\right) \approx 1.50\times10^{21}\,\mathrm{photons/s.}.$$

Behold ~ the answer is still not much different! In fact, if you plot B_λ(600 K) over our bandpass, you see that it is pretty flat, not changing by more than ≈ 19 % from end to end.



REMARK ABOUT THIS PROBLEM:

This multi-level solution goes well beyond anything you were expected you to do for your homework solution; it's just some things that you should be aware of. At any rate, if you gave an answer like the LEVEL ZERO one, without having considered the bandpass, that's OK for now. However, it is *very important* that you learn that all sensors **always** sample across a finite bandpass – the output you get is **never** representative of the sensor's input at a *single* wavelength. But always within some narrow bandpass that is usually defined by the transmission of the collecting optics (including a deliberate and specific filter usually) and the response function of the detector. These topics will all be covered in later chapters in the text.

2-11. *Stefan-Boltzmann Law.* Consider the Stefan-Boltzmann Law.

A. If the *temperature* of an object is doubled, by what factor is the total power per unit area emitted by the object increased?

B. By approximately what factor is the *power per unit area* emitted by an object in the microwave band increased when its temperature is doubled? [HINT: What approximation to the Planck function applies at "long" wavelengths?]

SUGGESTED ANSWER:

A. The point of this problem is to first note how the TOTAL power per unit area changes with temperature. From the Stephan-Boltzmann result,

$$B(T) = \int_0^\infty B_\lambda(\lambda, T)\, d\lambda = \sigma_{SB} T^4 \,,$$

we see that if $T \to 2T$, then $B(2T) = 16B(T)$

B. If we are only concerned with the microwave portion of the spectrum, then if we can approximate the Planck function with the Rayleigh-Jeans Limit, we have

$$B(T) \approx \int_{MICROWAVE} \frac{c_1}{c_2} \frac{T}{\lambda^4} d\lambda = \left(\frac{c_1}{c_2} \int_{MICROWAVE} \frac{d\lambda}{\lambda^4} \right) T$$

Since we see that power per unit area in the microwave band varies directly with the temperature, we have that when $T \to 2T$ the exitance goes as $B(2T) = 2B(T)$. This should dramatically emphasize to us that temperature increases are most apparent as IR/visible/UV power per unit area emissions.

2-13. *Planck Blackbody Radiation.* Consider the 2.00 – 3.50 μm SWIR band.

A. For which of the following steady state temperatures would an ideal object emit the most radiation in the band: 300 K, 900 K, 2700 K, or 8100 K? [HINT: remember what an integral represents on a graph.]

B. Estimate the power radiated in this band by a blackbody at these four temperatures.

C. Estimate the number of photons radiated per second these values represent.

☞ See spreadsheet Chapter 02 ~ Suggested Solutions (Odd).xlsx for solution. ☜

2-15. *Discussion on the Planck Function.* When you go to the hardware store to buy fluorescent light bulbs, you find that there are some which give a relaxing blue-white light that are called "cool white", and then there are some others which have a nice yellow-orange, even pinkish glow that are called "warm white." If the output of these bulbs was solely due to continuum radiation (it's really not), then based on your knowledge of the Planck function, WHAT IS WRONG HERE? (Should you tell the store manager?)

SUGGESTED ANSWER: From our knowledge of the Planck function, objects that are blackbodies increasingly emit more photons toward the blue end of the visible spectrum with increasing temperature. So physically, an object that appeared blue to us would actually be hotter than an object that appears red. If the fluorescent lamp bulbs were blackbody radiators, then they are labeled backwards from what our psycho-physiology tells us. Fluorescent lamps are not blackbodies, however, so there's really nothing wrong here – physics-ally speaking. Unless the store manager was a technoid like, it probably wouldn't get more than a blank stare, so don't bother bothering the poor overworked person.

2-17. ***Incandescent Light Bulb.*** The filament of a "100 W" light bulb is a thin tungsten wire about 0.500 mm in diameter (~ 24 AWG) and 3.40 cm long. When turned on, the filament temperature is approximately 2400 K. Assume it is a blackbody radiator.

A. At what wavelength is the maximum power per unit area per unit wavelength emitted?

B. What is the total *power per unit area* and the total *power* emitted?

C. Estimate how much power is emitted in the visible band (0.400 – 0.700 μm), and what fraction is this of the bulb's nominal rating?

D. How much power is emitted in a 2.00 nm wide bandpass centered at 0.555 μm?

E. For your last answer, how many photons per second is this?

☞ **See spreadsheet Chapter 02 ~ Suggested Solutions (Odd).xlsx for solution.** ☜

2-19. ***Temperature Measurement I.*** There are several ways to assign a temperature to a radiation emitter. Any such temperature is presumed to bear some relationship to the true temperature, but is probably not equal to it because all measurement methods assume ideal blackbody or graybody (constant emissivity) conditions. A simple method is to locate the wavelength of peak spectral exitance and deduce the corresponding blackbody temperature using Wien's displacement law (this assumes the ability to locate and measure the radiation peak).

A. If we measure that a certain emitter has its radiation peak at a wavelength of 1.50 μm, what is our estimate of its temperature (assuming it is a blackbody)?

B. If we determine that the object's graybody emissivity is 0.700, now what is our estimate of its temperature?

☞ **See spreadsheet Chapter 02 ~ Suggested Solutions (Odd).xlsx for solution.** ☜

2-21. ***Temperature Measurement III.*** In practice, real sensors do not measure *all* of the energy emitted, but only that in a finite bandpass. Nonetheless, a method similar to Problem 2-20 could be applied in a bandpass where a blackbody temperature is assigned to an (assumed) graybody radiator. That is, a graybody has an exitance equal to that of the blackbody (in the bandpass). Use the Planck function to estimate the brightness temperature of a source with emissivity 0.700 whose spectral exitance in a 4.00 μm bandpass in the thermal infrared (8.00 – 12.0 μm) is measured to be 1000 W·m^{-2}.

SUGGESTED ANSWER: The solution to this problem is somewhat more involved – but much more practical – than the previous problems. When a sensor observes what we will presume is a graybody source in a finite bandpass, its output is proportional to

$$M(\lambda_1,\lambda_2,T_{GB}) = \int_{\lambda_1}^{\lambda_2} M_\lambda(\lambda,T_{GB})d\lambda = \varepsilon_{GB}\int_{\lambda_1}^{\lambda_2} B_\lambda(\lambda,T_{GB})d\lambda$$

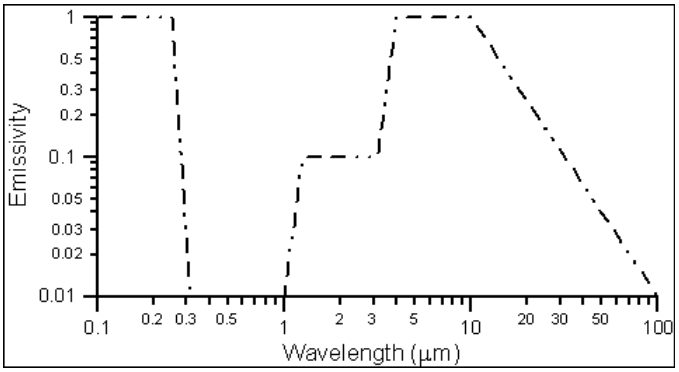
where λ_1 and λ_2 are the limits of the sensor's bandpass, ε_{GB} is the graybody emissivity, and T_{GB} is the actual temperature of the object.

In the spreadsheet, we have calculated the value of the integral of the blackbody function for assumed graybody temperatures from 250 to 950 K over the 8.00 – 12.0 μm bandpass, which gives about a two order-of-magnitude spread. We have then multiplied the integral by graybody emissivity values from 1.00 (a true blackbody) to 0.100, and have plotted the results. The plot actually turns the data around and shows us the temperature as a function of in-band exitance. As we would expect, the plot shows us that the object's exitance decreases as its emissivity decreases. [Note in the plot's legend that Excel flubbed trying to use the proper symbols!]

The meaning of these data (in the plot) are twofold: FIRST, when we **DO** know the emissivity of the object we are observing (as in the context of this problem), we enter the plot at the value of exitance we are observing and go up until we hit the emissivity value – when we read off the temperature of the object on the left axis. Here, 1000 W/m^2 gives us an approximate object temperature of 582 K (red diamond on plot). This is the object's actual temperature. But SECOND, if we **DON'T** know the object's emissivity, the only thing we can reasonably assume is that it is a blackbody (emissivity = 1). In that case, we again enter with our observed exitance, but only read up to the first curve (red triangle); we estimate that the temperature is 525 K.

Note that if we don't know the emissivity of an object, and we must assume that it is one, then the temperature we derive for the presumed blackbody will always be less than the object's *actual* temperature. That is, only a blackbody at a given temperature can emit the maximum amount of radiation, and any other object emitting the same amount must be at a higher temperature.

2-23. ***Emissivity.*** Given the plot at right for the emissivity of a peculiar alien material at 2727° C, sketch its spectral radiant exitance as a function of wavelength. <u>Note that the emissivity is plotted on log-log axes.</u> [HINT: Notice that the temperature is given in °C.]



☞ **See spreadsheet Chapter 02 ~ Suggested Solutions (Odd).xlsx for solution.** ☜

2-25. ***Aircraft Radiant Emission II.*** The exhaust plume behind a jet airplane cools quickly as it trails out behind, but it could be considered as a graybody with an average emissivity ≈ 0.400 and a mean temperature about 70% that of the tailpipe (~ 800 K).

A. If a typical exhaust plume is about 1.00 m in diameter and ~ 100 m long, what is its radiant power output in the 3.00 to 5.00 μm band?

B. What is the difference between this problem and the previous one (Problem 2-24)?

☞ **See spreadsheet Chapter 02 ~ Suggested Solutions (Odd).xlsx for solution to part A.** ☜

SUGGESTED ANSWER PART B: Evidently, the previous problem (2-24) calculates the power that appears to be emitted from a jet's tailpipe looking straight into it from the back, ignoring contributions or attenuation from the plume. This problem, however, calculates a power value looking at the exhaust plume from the side. The obvious geometrical difference is one of what is called "aspect" angle. Usually, the pointy end of an airplane is considered to be 0º aspect, and viewing from the side is 90º aspect. Looking up the tailpipe is then 180º aspect. The effect here, then, is that the last problem had us looking at a radiating area equal to that of the tailpipe's aperture ($\pi D^2/4 \approx 0.785\ m^2$); whereas this problem suggests that we "see" a radiating area equivalent to the cross-section of a cylindrical plume ($D \times L \approx 100\ m^2$).

There are some physical differences as well – the exhaust gasses cool (rather quickly too) as they trail behind the jet. This is for two reasons: *first*, the hot gasses are not in thermal equilibrium, so as they radiate they lose energy, hence cool; and *second*, they mix with the surrounding atmosphere and lose even more energy through molecular collisions and turbulence. From a remote sensing perspective, this means that we are not going to be looking at an object that is at a constant temperature. We could treat this by integrating over the plume – providing we know its temperature distribution from a combination of thermodynamics and fluid flow considerations – or we can just make an estimate like we do here that the plume has some average, effective value of temperature.

Furthermore, the hot (and cooling) exhaust gas is NOT a blackbody, but is semi-transparent, hence we need to apply some value of emissivity as well. Since we suggest that the emissivity at the tailpipe is 0.800 in the last problem, and we suppose that the gas has totally cooled by the time it has reached 100 m behind the plane where its emissivity would be the same as the ambient atmosphere (which we take to be ≈ 0), we just use an average value of 0.400. Note that this goes hand-in-hand with our presumption that the plume is semi-transparent and cools rapidly as a function of distance behind the plane.

2-27. ***Population of Atomic Energy States.*** If there are 10^6 atoms in Problem 2-26, and their temperature is 35,000 K, how many of them are in each energy state? How many are in each state if their temperature is 3,500 K? Assume that all the energy levels are non-degenerate.

☞ **See spreadsheet Chapter 02 ~ Suggested Solutions (Odd).xlsx for solution.** ☜

2-29. ***Hydrogen Electronic Energy States I.*** The electronic energy states of atomic hydrogen are given approximately by $E_n = -13.6/n^2$ eV where n is the "principle quantum number". Suppose that some hydrogen atoms are all excited from their ground state (n = 1) to the state n = 4.

A. What are the energy and wavelength of the photons necessary to induce this (upward) transition?

B. What are the energies and wavelengths of the possible photons that could be emitted by the excited hydrogen atoms as they (radiatively) relax back down to their ground state? (HINT: draw an energy level diagram.)

☞ **See spreadsheet Chapter 02 ~ Suggested Solutions (Odd).xlsx for solution.** ☜

2-31. ***Vibrational Energy States.*** Some carbon monoxide (CO) molecules are seen to emit 0.266 eV photons when they transition from their first vibrationally excited state (v = 1) to their ground state (v = 0). What would be the wavelength of photons emitted by the CO if it were to quantum leap from a v = 3 state to the v = 1 state?

☞ **See spreadsheet Chapter 02 ~ Suggested Solutions (Odd).xlsx for solution.** ☜

2-33. ***Population of States.*** Huge star-forming volumes of space are filled with atomic hydrogen gas. Ultraviolet light from actively forming stars (hot, young, blue OB stars) can fairly efficiently "pump" this gas into excited electronic energy states. Suppose that we observe that the light coming from one of these star-forming regions consists entirely of Hα (656.3 nm) and Hβ (486.1 nm) radiation, and that the Hα is twice as "bright" as the Hβ. (That is, there are twice as many Hα photons as Hβ.) Use this information to estimate the temperature of the atomic hydrogen gas.

SUGGESTED ANSWER: The famous Hα (red) transition is an emission from a hydrogen atom quantum leaping from its <u>second excited state</u> (n = 3) to its <u>first excited state</u> (quantum number n = 2), and the Hβ transition is from the <u>third excited state</u> (n = 4) to the <u>first</u>. Problem 2-29 reminds us that the energy levels of the hydrogen atom are given by $E_n = {-13.6}/{n^2}\ \text{eV}$. For the second and third excited states, these energies are E_3 = –1.51 eV and E_4 = –0.850 eV, respectively. (Remember that the negative energy value means bound state in this case.)

Without knowing any other details of these emissions (oscillator strengths, branching ratios, etc.), we can simply infer from the problem statement that the populations of the upper energy states for the Hα and Hβ transitions are in the ratio of two to one, i.e., $N_3 = 2N_4$, just like the observed line brightness's from the gas cloud. Since the population of an energy state is given by $N_i = \frac{N_T}{Z} g_i\, e^{-\Delta E_i / k_B T}$, we can calculate $N_3 = \frac{N_T}{Z} g_3\, e^{-\Delta E_3 / k_B T}$ and $N_4 = \frac{N_T}{Z} g_4\, e^{-\Delta E_4 / k_B T}$ for our two emitting states. Assuming that $g_3 = g_4$, we can take the ratio and simplify:

$$\frac{N_3}{N_4} = \frac{2N_4}{N_4} = 2 = \frac{\frac{N_T}{Z} g_3\, e^{-\Delta E_3 / k_B T}}{\frac{N_T}{Z} g_4\, e^{-\Delta E_4 / k_B T}} = \frac{e^{-\Delta E_3 / k_B T}}{e^{-\Delta E_4 / k_B T}} = e^{(\Delta E_4 - \Delta E_3)/k_B T}$$

Substituting the definition for $\Delta E_i = E_i - E_G$, where E_G means ground state energy ($E_G = E_1$ = –13.6 eV for hydrogen), we have:

$$2 = e^{([E_4 - E_1] - [E_3 - E_1])/k_B T} = e^{(E_4 - E_3)/k_B T}$$

Taking the natural logarithm, this can be immediately solved for our estimate of the temperature:

$$\boxed{T = \frac{E_4 - E_3}{(\ln 2) k_B} = \frac{(-0.850\,\text{eV} + 1.51\,\text{eV})(1.60\times 10^{-19}\ \text{J}/\text{eV})}{(0.693)(1.38\times 10^{-23}\ \text{J}/\text{K})} = 11{,}000\ \text{K}}$$