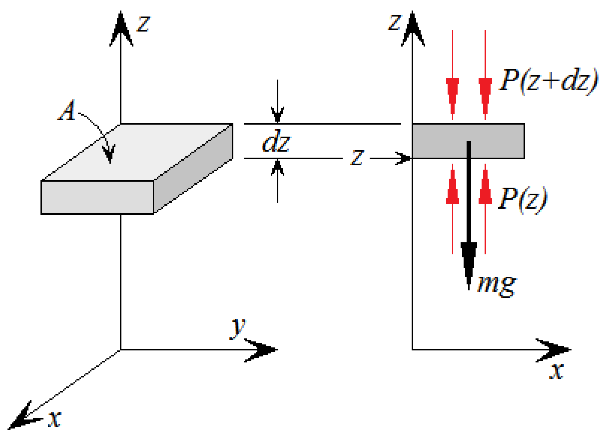
APPENDIX 5-3: Physical Derivation of the Exponential Dependence

The physical derivation of the hydrostatic equation (**Equation 5-2**) mentioned in **Footnote 11** goes like this ~



Consider a "box" of air at altitude z (bottom side) having a height dz and an area A. For the box to be static, the upward and downward forces must balance. The upward force is just the pressure acting on the area of the box, but the downward force is the sum of downward pressure and the weight of air in the box:

$$P(z)A = P(z+dz)A + mg \tag{1}$$

Since dz is taken to be small, let the pressure at the top of the box* be $P(z)+dP$. Also, let the weight of air in the box be

$$mg = \langle m \rangle g\, n\, A dz \tag{2}$$

where $\langle m \rangle$ is the mass per molecule,

n is the number of molecules per unit volume, and
Adz is the volume of the box.

Now,

$$\begin{aligned} P(z)A &= P(z)A + dPA + \langle m \rangle g\, n\, A dz \\ 0 &= dPA + \langle m \rangle g\, n\, A dz \\ dP &= -\langle m \rangle g\, n\, A dz. \end{aligned} \tag{3}$$

Next, assume the ideal gas law with constant temperature applies in the form $P = nk_BT$ (see **Footnote 8**), such that

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* Since pressure decreases going upward, dP is inherently negative. The mathematics will show this.

$$dP = k_B T dn \tag{4}$$

Substituting,

$$k_B T dn = -\langle m \rangle g \, n \, dz \tag{5}$$

To solve this differential equation, separate variables and integrate:

$$\int \frac{dn}{n} = -\int \frac{\langle m \rangle g}{k_B T} dz$$

$$\ln n + C = -\frac{\langle m \rangle g}{k_B T} z. \tag{6}$$

It is most convenient to represent the constant of integration as $C = -\ln n_0$ and the group of constants on the right hand side as $\frac{\langle m \rangle g}{k_B T} = \frac{1}{\hat{H}}$. Thus,

$$\ln n - \ln n_0 = \ln \frac{n}{n_0} = -\frac{1}{\hat{H}} z$$

$$\frac{n}{n_0} = e^{-\frac{1}{\hat{H}} z}, \tag{7}$$

or

$$n = n_0 e^{-z/\hat{H}} \tag{8}$$

It can easily be shown that $\hat{H}$ has the dimensions of length, and is known as the scale height.